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CONCENTRATED RING LOADINGS IN A FULL SPACE OR HALF SPACE: SOLUTIONS FOR TRANSVERSE ISOTROPY AND ISOTROPY

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Abstract—In this analysis solutions for concentrated ring loading in a transversely isotropic full space or half space are found. The elastic field is derived by integrating the known fundamental point force solutions along a circular ring. Three cases of ring loading are considered. Taking the z axis along the material axis (the $z = 0$ plane is the isotropic plane), the first case studied is when the ring load is in the z direction, referred to as normal loading. The other two cases are shear ring loads directed in the plane of isotropy. In the first instance, the ring load is unidirectional and independently applied in either the x or y direction. The second case considers the axisymmetric radial and axisymmetric torsional ring loads. The solution for ring loading applied to the surface of a half space is first obtained. Subsequently the solutions for ring loading in a full space and buried ring loads in a half space are found. In all cases the elastic displacement and stress fields are evaluated in terms of closed form expressions containing complete elliptic integrals of the first, second and third kinds. An interesting feature of the full space solution is that the potential function and its radial derivatives exhibit a cylindrical discontinuity for negative z values. However, it is shown that these discontinuous functions do provide continuous displacement and stress fields. A limiting form of the solutions for transverse isotropy also provides the corresponding results for isotropic materials.

1. INTRODUCTION

Fundamental point force solutions have played an important role in the application of linear elasticity to solve problems of practical interest. In particular, the point force solution by Kelvin for a full space, the solutions by Boussinesq and Cerruti for point loading on the surface of a half space (see for example, Love, 1927) and the Mindlin (1936) solution for a point force buried in a half space have found many uses in contact mechanics (Johnson, 1985), micromechanics (Mura, 1982) and Boundary Element Methods. To a large degree, application of these solution usually involves integration. For example, elastic fields resulting from concentrated contacts can be found by integrating the Boussinesq and Cerruti solutions (Johnson, 1985). Elastic fields in a full space or half space caused by inclusions, inhomogeneities or cracks can be obtained by integrating derivatives of the Kelvin or Mindlin solutions (Mura, 1982). In studying composite materials, the bimaterial Green's function given by Rongved (1955) has found similar applications.

Integration of a point force solution around the circumference of a circle leads to what is presently termed a ring load. Such a solution may have many uses. If a surface or interior load is given over a circular area then the elastic field can be obtained by integrating the appropriate point force solution. If the load intensity does not have an angular dependence over the circular area, the elastic field can be obtained by multiplying the ring loading solution by the radial load distribution and integrating in the radial direction only. The solution for uniform pressure applied to a circular area on the surface of a half space solved by Love (1929) could have been obtained in this manner. Another example, displaying the usefulness of the ring loading solutions, is provided by Hasegawa *et al.* (1992b, 1993). In these papers the axisymmetric ring loads were integrated to obtain closed form expressions for the elastic field caused by a uniform eigenstrain prescribed in a right circular cylinder embedded in a full space or a half space.

A common thread among the solutions noted above is that they are for isotropic materials. For non-isotropic materials closed form solutions to the point force Green's functions are very limited. The exception is when the material is transversely isotropic and the isotropic planes are parallel to the surface or interface if present. In this case the solutions have a strong resemblance to their isotropic counterparts. The solutions are more complicated in that five elastic constants are involved. However, the elastic fields in many cases have a simpler form. The first point force Green's function for an infinite body was given by Elliot (1948) when the force was perpendicular to the isotropic plane. Pan and Chou (1976) noted other previous investigations and gave a solution for a general force. Probably Shield (1951) was the first to give the solution to a force buried in a half space when the force was perpendicular to the surface. Fabrikant (1970) and Pan and Chou (1979) note some other previous half space solutions and give a solution for a general force near the surface of a transversely isotropic half space. Fabrikant (1989) has considered many mixed boundary value problems for the transversely isotropic half space and has developed a very convenient form to the full space or half space potentials and the general expressions for the elastic field. This notation is adopted here.

In the present study, the potentials provided by Fabrikant (1989) for a point force in a full space or on the surface of a half space are used. The potentials for a point force buried in a transversely isotropic half space are derived in Appendix E using the present notation. These potentials are integrated around a circle to obtain the solution for ring loading. Three distinct loading cases are considered. In the first instance the ring load is directed along the material axis (presently denoted as z). This is termed axisymmetric normal loading since it acts normal to the isotropic plane and perpendicular to the half space surface if present. The second case is a unidirectional shear ring load in either the x or y direction which acts in the plane of isotropy. The third loading is axisymmetric radial and axisymmetric torsional ring loading, again in the isotropic plane.

Some of the ring loads considered here have been evaluated previously. Kermanidis (1975) used the elastic reciprocal theorem to evaluate the elastic displacement fields for ring loading in an isotropic full space. He considered the axisymmetric cases of normal, radial and torsional loads. His results were given in terms of complete elliptic integrals of the first and second kinds. Similar results have recently been derived by Hasegawa *et al.* (1992a) using Hankel and Fourier transforms. They evaluated the elastic field for analogous ring loads applied near a bimaterial interface. Their results were given in terms of two Legendre functions and the complete elliptic integral of the third kind. For similar materials their results are in agreement with those of Kermanidis (1975). For a transversely isotropic non-homogeneous material, Erguvan (1987, 1988) has considered the axisymmetric torsional ring loading case only. Hasegawa and Watanabe (1995) have evaluated the displacement fields for axisymmetric ring loading applied on the surface of a transversely isotropic half space. Hasegawa and Ariyoshi (1995) considered a related problem for a transversely isotropic full space. The present authors are unaware of any ring loading solution for unidirectional shear traction which is one of the cases considered in this study.

As noted above, the ring loading solutions will be evaluated by directly integrating the potentials for the point force solutions around the circumference of a circle. Although the derivatives of the point force potentials are continuous functions, it is shown that integrating them around a circle leads to functions with a discontinuity along the cylindrical shell $\rho = a$, $z < 0$, where $\rho = a$ is the radius of loading. If the ring loading is applied to the surface of the half space $z > 0$, this discontinuity is of no consequence. For full space problems or ring loading buried in a half space, this discontinuity enters into the solution. The nature of the solution for transverse isotropy leads to the elastic field being written as the sum of two or three terms. It is shown that although each term in the summation is discontinuous, the discontinuities cancel in the summation process leading to a continuous elastic field. Here, continuous, refers to the regions $z > 0$ or $z < 0$. On the plane $z = 0$ the elastic field may still contain a discontinuity as one passes from inside to outside the ring of loading. However, the elastic field is continuous as one passes across the $z = 0$ plane, say going from $z > 0$ to $z < 0$, either inside or outside the ring of loading. The direct integration of the potential functions leads to their evaluation in terms of complete elliptic integrals of the first, second and third kinds. The form of the elliptic integral parameter naturally occurring in this process leads to an elliptic integral of the third kind which has a discontinuity along the entire cylinder $\rho = a$. A better form to the solution is obtained using well known transformation formulas for the complete elliptic integrals of the first and second kind along with a similar but more complicated transformation formula for the complete elliptic integral of the third kind recently derived by Hanson and Puja (1996a). This form allows the discontinuity in the derivatives of the potential functions for $\rho = a$, $z < 0$ to be isolated in a simple form. The continuous part of these evaluations is written in terms of the function $I(\mu, \nu; \lambda)$ introduced by Eason *et al.* (1955). This function is an infinite integral involving products of Bessel functions of integer order, an exponential and a power. For various integer values of μ, ν and λ, $I(\mu, \nu; \lambda)$ was recently re-evaluated in a more convenient form by Hanson and Puja (1996b). These results are used extensively in the present evaluations for the elastic fields.

The solution for transversely isotropic materials has an additional complication not occurring in isotropic results. This is caused by the z coordinate being scaled by three elastic parameters, two of which may be complex conjugate for some materials. Thus, any function (such as complete elliptic integrals) which occur in the expressions for the elastic field must be evaluated for complex values of their parameters. The first several sections of this paper pay particular attention to this issue. Then the different ring loadings are considered when applied to the surface of a half space, in a full space and buried in a half space. The details of the integration processes needed for each different geometry are deferred to appendices for the interested reader. Finally, the isotropic solutions for ring loading on the surface of a half space, in a full space or buried in a half space are obtained by taking a limiting form of the solutions for transverse isotropy. The details of this are also provided in the appendices.

2. POTENTIAL FUNCTIONS FOR TRANSVERSE ISOTROPY

A potential function formulation for transverse isotropy was first given by Elliot (1948). The notation of Fabrikant (1989) is presently adopted. The stress strain relations in Cartesian components are given below with the z axis taken as the axis of material symmetry

$$\begin{aligned}
\sigma_{xx} &= A_{11}\frac{\partial u}{\partial x} + (A_{11} - 2A_{66})\frac{\partial v}{\partial y} + A_{13}\frac{\partial w}{\partial z},\\
\sigma_{yy} &= (A_{11} - 2A_{66})\frac{\partial u}{\partial x} + A_{11}\frac{\partial v}{\partial y} + A_{13}\frac{\partial w}{\partial z},\\
\sigma_{zz} &= A_{13}\frac{\partial u}{\partial x} + A_{13}\frac{\partial v}{\partial y} + A_{33}\frac{\partial w}{\partial z},\\
\tau_{xy} &= A_{66}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),\\
\tau_{xz} &= A_{44}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right),\\
\tau_{yz} &= A_{44}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right).
\end{aligned} \tag{1}$$

Here, u, v and w are the displacements in the x, y and z directions and A_{11}, A_{13}, A_{33}, A_{44} and A_{66} are the elastic constants.

The solution of the equilibrium equations in terms of three potential functions F_1, F_2 and F_3 is given by Fabrikant (1989) in the form

$$u^c = \Lambda(F_1+F_2+iF_3), \quad w = m_1\frac{\partial F_1}{\partial z}+m_2\frac{\partial F_2}{\partial z}, \tag{2}$$

with i being the complex number, $i = \sqrt{(-1)}$, m_1 and m_2 are constants defined below, and u^c is the complex displacement $u^c = u+iv$. The operator Λ and the operator Δ used subsequently are given as

$$\Lambda = \frac{\partial}{\partial x}+i\frac{\partial}{\partial y} = e^{i\phi}\frac{\partial}{\partial \rho}+\frac{ie^{i\phi}}{\rho}\frac{\partial}{\partial \phi}, \quad \Delta = \Lambda\bar{\Lambda} = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}. \tag{3}$$

The functions F_j satisfy the relations

$$\Delta F_j+\gamma_j^2\frac{\partial^2 F_j}{\partial z^2} = 0, \quad j = 1,2,3, \tag{4}$$

where γ_j are also constants. The constant γ_3 is given as $\gamma_3^2 = A_{44}/A_{66}$ while $\gamma_j^2 = n_j, j = 1, 2$ and n_j are the two (real or complex conjugate) roots of the quadratic equation

$$A_{11}A_{44}n_j^2+[A_{13}(A_{13}+2A_{44})-A_{11}A_{33}]n_j+A_{33}A_{44} = 0. \tag{5}$$

The constants m_j are related to γ_j as

$$m_j = \frac{A_{11}\gamma_j^2-A_{44}}{A_{13}+A_{44}} = \frac{(A_{13}+A_{44})\gamma_j^2}{A_{33}-\gamma_j^2A_{44}}, \quad j = 1,2. \tag{6}$$

Using the stress combinations in Cartesian or cylindrical coordinates $\sigma_1 = \sigma_{xx}+\sigma_{yy} = \sigma_{\rho\rho}+\sigma_{\phi\phi}$, $\sigma_2 = \sigma_{xx}-\sigma_{yy}+2i\tau_{xy} = e^{2i\phi}(\sigma_{\rho\rho}-\sigma_{\phi\phi}+2i\tau_{\rho\phi})$ and $\tau_z = \tau_{xz}+i\tau_{yz} = e^{i\phi}(\tau_{\rho z}+i\tau_{\phi z})$, the stress field can be written in the following form

$$\sigma_1 = 2A_{66}\frac{\partial^2}{\partial z^2}\{[\gamma_1^2-(1+m_1)\gamma_3^2]F_1+[\gamma_2^2-(1+m_2)\gamma_3^2]F_2\},$$

$$\sigma_2 = 2A_{66}\Lambda^2[F_1+F_2+iF_3],$$

$$\sigma_{zz} = A_{44}\frac{\partial^2}{\partial z^2}[\gamma_1^2(1+m_1)F_1+\gamma_2^2(1+m_2)F_2],$$

$$\tau_z = A_{44}\Lambda\frac{\partial}{\partial z}[(1+m_1)F_1+(1+m_2)F_2+iF_3]. \tag{7}$$

At this point some discussion as to the nature of the above solution is in order. It will be seen below that the three potential functions are written in terms of a single function as $F_j = c_jF(z_j)$ where $z_j = z/\gamma_j$ and c_j are constants that depend on γ_j and m_j. From eqn (5) the two roots n_1 and n_2 are either both real or complex conjugates. Hence m_1 and m_2 are also both real or complex conjugates. The quantities γ_1 and γ_2 were introduced by Fabrikant (1989) to denote $\sqrt{n_1}$ and $\sqrt{n_2}$, respectively. Since γ_3 is always a real positive number, real positive values for n_1 and n_2 (and hence m_1, m_2, γ_1 and γ_2) will give real displacement and stress fields. On the other hand, complex conjugates n_1 and n_2 imply complex conjugates m_1 and m_2 and thus γ_1 and γ_2 must also be complex conjugates for the elastic field to be a real quantity (Elliot, 1948). This forces the location of the branch cut in the complex plane for the square root function. Therefore the polar angle θ in the complex plane must be measured as $-\pi \leqslant \theta < \pi$ for the complex numbers n_1 and n_2. Thus $\gamma_1 = \sqrt{n_1}$ and $\gamma_2 = \sqrt{n_2}$ are then complex conjugate with a positive real part. Since m_1 and m_2 satisfy the relation $m_1m_2 = 1$ (Fabrikant, 1989), complex conjugates m_1 and m_2 imply they each have a modulus of unity.

The different cases can be summarized as follows. If n_1 and n_2 are positive real numbers then γ_1 and γ_2 are the positive real square roots. If n_1 and n_2 are complex conjugates, then γ_1 and γ_2 are their complex conjugate square roots with positive real parts. The case of one real positive root and one real negative root cannot occur (see Appendix F) and thus the only other possibility is if n_1 and n_2 are both real and negative ($\theta = -\pi$). In this case their square roots would be negative imaginary numbers ($\theta/2 = -\pi/2$). Hence, both γ_1 and γ_2 would be complex but not conjugate and real solutions for the elastic field would not result. This conclusion appears to be true for any definition of the branch of the square root function. It is shown in Appendix F (for axisymmetric deformation see also Lekhnitskii, 1960) that this case cannot occur based on certain well known restrictions imposed on the elastic constants.

The final point that needs to be clarified concerns the quantities $z_j = z/\gamma_j$. Since γ_3 is always positive real, z_3 is always real. Hence the sign of z_3 depends on the sign of z. Similar reasoning also applies to z_1 and z_2 if γ_1 and γ_2 are positive real. If γ_1 and γ_2 are complex conjugate (and thus have a positive real part), then $1/\gamma_1$ and $1/\gamma_2$ also have positive real parts. Thus the sign of Re $\{z_1\}$ and Re $\{z_2\}$ depends only on the sign of z. If one considers a half space problem $z > 0$, then z_3 is a positive real number and z_1, z_2 are either positive real or have positive real parts. If a full space problem is considered, the above reasoning also applies to the region $z > 0$. In the region $z < 0$, z_3 is a negative real number and z_1, z_2 are either negative real or have negative real parts. One may conclude in all cases that the sign of Re $\{z_j\}$ ($j = 1, 2, 3$) is the same as the sign of z itself.

3. ELLIPTIC INTEGRALS AND SOLUTION PARAMETERS

The solutions to all of the problems considered in this paper will be written in terms of complete elliptic integrals of the first, second and third kinds. These are denoted as $\mathbf{F}(k)$, $\mathbf{E}(k)$ and $\Pi(n, k)$ respectively. They are given in standard form as (Gradshteyn and Ryzhik, 1980)

$$\mathbf{F}(k) = \int_0^1 \frac{\mathrm{d}x}{(1-x^2)^{1/2}(1-k^2x^2)^{1/2}} = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{(1-k^2\sin^2\theta)^{1/2}}, \tag{8}$$

$$\mathbf{E}(k) = \int_0^1 \frac{(1-k^2x^2)^{1/2}\,\mathrm{d}x}{(1-x^2)^{1/2}} = \int_0^{\pi/2} (1-k^2\sin^2\theta)^{1/2}\,\mathrm{d}\theta, \tag{9}$$

$$\Pi(n,k) = \int_0^1 \frac{\mathrm{d}x}{(1-nx^2)(1-x^2)^{1/2}(1-k^2x^2)^{1/2}} = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{(1-n\sin^2\theta)(1-k^2\sin^2\theta)^{1/2}}, \tag{10}$$

where k is called the modulus, $k' = (1-k^2)^{1/2}$ is the complementary modulus and n is the parameter. Note that $0 < k, k', n < 1$ must hold for real positive values of these quantities. The solutions will contain the two parameters $l_1(a)$, $l_2(a)$ given as

$$\begin{aligned} l_1(a) &= \tfrac{1}{2}\{[(\rho+a)^2+z^2]^{1/2} - [(\rho-a)^2+z^2]^{1/2}\}, \\ l_2(a) &= \tfrac{1}{2}\{[(\rho+a)^2+z^2]^{1/2} + [(\rho-a)^2+z^2]^{1/2}\}. \end{aligned} \tag{11}$$

These parameters, first introduced by Fabrikant (1989), allow the three-dimensional distance between the point $(a, 0, 0)$ on the surface and the interior point (ρ, ϕ, z) to be written in two-dimensional form. That is

$$\rho^2+a^2+z^2-2a\rho\cos(\phi) = l_1^2(a)+l_2^2(a)-2l_1(a)l_2(a)\cos(\phi), \tag{12}$$

where

$$l_1^2(a)+l_2^2(a) = \rho^2+a^2+z^2, \quad l_1(a)l_2(a) = a\rho. \tag{13}$$

Note that it is easy to see $l_1(a) < l_2(a)$ while it can also be shown that $l_1(a) < \rho$. Both inequalities are easily verified for real values of z. For the solutions in this paper, the parameters k and n in eqns (8)–(10) will be given as

$$k = \frac{l_1(a)}{l_2(a)}, \quad n = \frac{l_1^2(a)}{\rho^2} = \frac{a^2}{l_2^2(a)}. \tag{14}$$

Note that $k = 1$ when $z = 0$, $\rho = a$ while $n = 1$ at $z = 0$, $\rho < a$.

The actual solutions in this paper will depend on $l_1(a)$, $l_2(a)$, k and n as above but with z replaced by z_j, $j = 1, 2, 3$. In this case they will be denoted as $l_{1j}(a)$, $l_{2j}(a)$, k_j and n_j. If z_j is real, then $l_{1j}(a)$, $l_{2j}(a)$, k_j and n_j are real and satisfy the inequalities above. Now consideration is given to the case when z_1 and z_2 are complex conjugates. The same branch of the complex square root function will be used as above. That is, the square root of a complex number will have a positive real part ensuring square roots of complex conjugate numbers will themselves be complex conjugate. In this case it is apparent that $\mathrm{Re}\{[(\rho+a)^2+z_j^2]^{1/2}\}$ and $\mathrm{Re}\{[(\rho-a)^2+z_j^2]^{1/2}\}$ are positive. Thus $\mathrm{Re}\{l_{2j}(a)\} > 0$, $j = 1, 2$.

Now it will be established that the modulus of k_j [denoted as $|k_j|$] for $j = 1, 2$ is less than unity. Since z_j^2 can be any complex number, first assume $\mathrm{Im}\{z_j^2\} > 0$. In this case one can write

$$[(\rho+a)^2+z_j^2]^{1/2} = e+ib, \quad e, b > 0, \quad [(\rho-a)^2+z_j^2]^{1/2} = c+id, \quad c, d > 0,$$

$$k_j = \frac{l_{1j}(a)}{l_{2j}(a)} = \frac{(e-c)+i(b-d)}{(e+c)+i(b+d)}, \quad |k_j| = \frac{[(e-c)^2+(b-d)^2]^{1/2}}{[(e+c)^2+(b+d)^2]^{1/2}} < 1. \tag{15}$$

In a similar manner it can be shown that $|k_j| < 1$ when $\mathrm{Im}\{z_j^2\} < 0$.

It can also be easily shown that $|n_j| < 1$ if $\rho > a$. To see this it is first noted that $l_{1j}(a)l_{2j}(a) = a\rho = |l_{1j}(a)|\,|l_{2j}(a)|$ and this product is always a real number. Using eqn (15) above one has

$$|k_j| = \frac{|l_{1j}(a)|}{|l_{2j}(a)|} = \frac{|l_{1j}(a)|^2}{|l_{1j}(a)|\,|l_{2j}(a)|} = \frac{|l_{1j}(a)|^2}{a\rho} < 1. \tag{16}$$

Using this last result, $|n_j|$ can be given as

$$|n_j| = \frac{|l_{1j}(a)|^2}{\rho^2} = \frac{|l_{1j}(a)|^2}{a\rho}\frac{a}{\rho} < 1, \tag{17}$$

where the inequality holds if $\rho > a$. It remains to establish this result for $\rho < a$.

Some comments pertaining to the elliptic integrals defined in eqns (8)–(10) are also necessary. They are denoted as $\mathbf{F}(k_j)$, $\mathbf{E}(k_j)$ and $\Pi(n_j, k_j)$ in the solutions for the elastic field since they depend on z_j. These will be complex valued functions when z_1 and z_2 are complex conjugate. Note that $\mathrm{Re}\{1-k_j^2\sin^2\theta\} > 0$ since $|k_j| < 1$. To determine the elastic field at a general point these integrals must be numerically evaluated. Thus the square root $(1-k_j^2\sin^2\theta)^{1/2}$ must also be chosen as above with a positive real part such that $\mathbf{F}(k_1)$, $\mathbf{E}(k_1)$, $\Pi(n_1, k_1)$ and $\mathbf{F}(k_2)$, $\mathbf{E}(k_2)$, $\Pi(n_2, k_2)$ are complex conjugate.

Some final results which will be subsequently needed are now discussed. Since $|k_j| < 1$, then

$$\text{Re}\{1+k_j\} > 0. \tag{18}$$

Also consider the difference $l_{2j}(a)-l_{1j}(a)$ given as

$$l_{2j}(a)-l_{1j}(a) = l_{2j}(a)[1-k_j] = [(\rho-a)^2+z_j^2]^{1/2}. \tag{19}$$

Taking the limit $\rho \to a$ in the above equation provides

$$\lim_{\rho\to a}[l_{2j}(a)-l_{1j}(a)] = \lim_{\rho\to a} l_{2j}(a)[1-k_j] = [z_j^2]^{1/2}. \tag{20}$$

Using the branch of the complex square root defined above, it can be shown that

$$[z_j^2]^{1/2} = z_j, \quad \text{Re}\{z_j\} > 0; \quad [z_j^2]^{1/2} = -z_j, \quad \text{Re}\{z_j\} < 0. \tag{21}$$

If z_j is real, the above degenerates to the obvious result $[z_j^2]^{1/2} = |z_j|$, where in this case the vertical bars indicate absolute value. Using the above result, it is also easy to verify that

$$[l_{2j}^2(a)]^{1/2} = l_{2j}(a), \quad [(1+k_j)^2]^{1/2} = (1+k_j), \tag{22}$$

since they both have positive real parts as shown above.

4. TRANSFORMATION FORMULAE FOR THE COMPLETE ELLIPTIC INTEGRALS

Several transformation formulae for complete elliptic integrals will be useful. For the complete elliptic integrals of the first and second kinds, the well known formulae are (Gradshteyn and Ryzhik, 1980)

$$\mathbf{F}\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k)\mathbf{F}(k), \quad \mathbf{E}\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{[2\mathbf{E}(k)-(1-k^2)\mathbf{F}(k)]}{(1+k)}. \tag{23}$$

A formula for the complete elliptic integral of the third kind was derived by Hanson and Puja (1996a) which can be put in the form

$$(a-\rho)\Pi\left(p,\frac{2\sqrt{k}}{1+k}\right) = \frac{\pi l_2(a)(1+k)(a+\rho)}{2z}\{1+\text{sgn}(a-\rho)\}$$

$$+(1+k)(a+\rho)\{\mathbf{F}(k)-2\Pi(n,k)\}, \quad p = \frac{4a\rho}{(a+\rho)^2}. \tag{24}$$

The above equations were previously used by Hanson and Puja (1996a,b) when z and hence, k, n were real quantities, and in particular $z > 0$. Equations (23) are even functions of z and hence also valid for $z < 0$. The first term on the right side of eqn (24) is an odd function of z while all the other terms in this equation are even functions. Thus, if z is replaced by absolute value of z in the denominator of the first term on the right side, eqn (24) is also valid for $z < 0$. Also note that $(a-\rho)\Pi[p,(2\sqrt{k}/1+k)]$ has a discontinuity along the cylinder $\rho = a$ (where $p = 1$), caused by the divergence of this integral.

The above equations are also needed when z is the complex quantity z_j. Equations (23) are valid for complex valued k_j but eqn (24) is not. To fix this equation the discontinuity in $\Pi[p,(2\sqrt{k}/1+k)]$ must be analyzed. From Byrd and Friedman (1971) the behavior as $p \to 1$ is

$$\Pi(\alpha^2, t) = \mathbf{F}(t) - \frac{\mathbf{E}(t)}{1-t^2} + \frac{\pi[2-t^2-t^2\alpha^2]}{4(1-t^2)^{3/2}(1-\alpha^2)^{1/2}} + O(1-\alpha^2). \tag{25}$$

Substituting $\alpha^2 = p = 4a\rho/(a+\rho)^2$ and $t = 2\sqrt{k}/(1+k)$ and simplifying leads for $\rho \to a$ to

$$\operatorname*{Lim}_{\rho\to a}(a-\rho)\Pi\left(p, \frac{2\sqrt{k}}{1+k}\right) = \frac{\pi(1+k)(a+\rho)}{2(1-k)}\operatorname{sgn}(a-\rho) + O(a-\rho), \tag{26}$$

where k is evaluated at $\rho = a$. Now if k is replaced by k_j then from eqn (20) the above result becomes

$$\operatorname*{Lim}_{\rho\to a}(a-\rho)\Pi\left(p, \frac{2\sqrt{k_j}}{1+k_j}\right) = \frac{\pi l_{2j}(a)(1+k_j)(a+\rho)}{2[z_j^2]^{1/2}}\operatorname{sgn}(a-\rho) + O(a-\rho). \tag{27}$$

Therefore eqn (24) can be modified to complex values as

$$(a-\rho)\Pi\left(p, \frac{2\sqrt{k_j}}{1+k_j}\right) = \frac{\pi l_{2j}(a)(1+k_j)(a+\rho)}{2[z_j^2]^{1/2}}\{1+\operatorname{sgn}(a-\rho)\}$$
$$+(1+k_j)(a+\rho)\{\mathbf{F}(k_j) - 2\Pi(n_j, k_j)\}, \tag{28}$$

where the square root is chosen as in eqn (21). Letting $\rho \to a$ in this last equation produces another interesting result

$$\operatorname*{Lim}_{\rho\to a}\{\mathbf{F}(k_j) - 2\Pi(n_j, k_j)\} = -\frac{\pi l_{2j}(a)}{2[z_j^2]^{1/2}}, \tag{29}$$

which can also be obtained from eqn (26) of Hanson and Puja (1995b).

5. RING LOADING ON THE SURFACE OF A HALF SPACE

Consider the transversely isotropic half space $z > 0$ shown in Fig. 1, where the plane of the surface is an isotropic plane. Using cylindrical coordinates (ρ, ϕ, z), a point force is

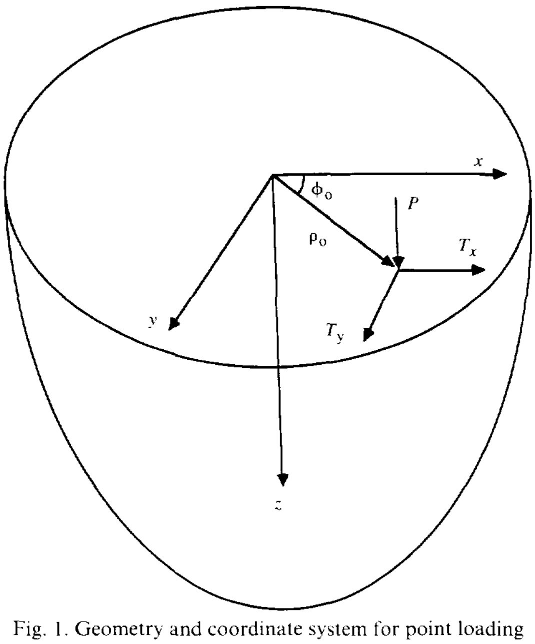

Fig. 1. Geometry and coordinate system for point loading

applied on the surface at ρ_0, ϕ_0 with components T_x, T_y and P in the x, y and z directions, respectively. The potential functions for these fundamental point force solutions were put in a very convenient form by Fabrikant (1989). For the point normal force P, the potentials are

$$F_1(\rho,\phi,z;\rho_0,\phi_0) = \frac{PH\gamma_1}{(m_1-1)}\ln[R_1+z_1]$$

$$F_2(\rho,\phi,z;\rho_0,\phi_0) = \frac{PH\gamma_2}{(m_2-1)}\ln[R_2+z_2]$$

$$F_3(\rho,\phi,z;\rho_0,\phi_0) = 0 \tag{30}$$

with

$$R_j^2 = \rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z_j^2, \quad z_j = \frac{z}{\gamma_j}, \quad j = 1,2,3, \tag{31}$$

and the real constant H is defined as

$$H = \frac{(\gamma_1+\gamma_2)A_{11}}{2\pi(A_{11}A_{33}-A_{13}^2)}. \tag{32}$$

The potentials for point shear loading are given as

$$F_1(\rho,\phi,z;\rho_0,\phi_0) = \frac{H\gamma_1}{(m_1-1)}\frac{\gamma_2}{2}(T\bar{\Lambda}+\bar{T}\Lambda)\chi(z_1)$$

$$F_2(\rho,\phi,z;\rho_0,\phi_0) = \frac{H\gamma_2}{(m_2-1)}\frac{\gamma_1}{2}(T\bar{\Lambda}+\bar{T}\Lambda)\chi(z_2)$$

$$F_3(\rho,\phi,z;\rho_0,\phi_0) = i\frac{\gamma_3}{4\pi A_{44}}(T\bar{\Lambda}-\bar{T}\Lambda)\chi(z_3) \tag{33}$$

where $T = T_x+iT_y$, an overbar indicates complex conjugation and the function $\chi(z_j)$ is

$$\chi(z_j) = z_j\ln[R_j+z_j]-R_j, \quad j = 1,2,3. \tag{34}$$

5.1. *Ring normal loading*

First consider an axisymmetric ring normal load on the surface located at a radius of $\rho = a$ with a density (force per unit circumferential length) of Q as shown in Fig. 2. To obtain the potentials from eqn (30) set $\rho_0 = a$, replace the force P with $Qa\,\mathrm{d}\phi_0$ and integrate the result from $0 < \phi_0 < 2\pi$. The potentials become

$$F_j(\rho,\phi,z) = \frac{H\gamma_jQa}{(m_j-1)}\psi(\rho,z_j), \quad j = 1,2,$$

$$\psi(\rho,z) = \int_0^{2\pi}\ln[R+z]\,\mathrm{d}\phi_0, \quad R^2 = \rho^2+a^2-2a\rho\cos(\phi_0)+z^2, \tag{35}$$

where ϕ has been set equal to zero since it can be shown that the integral is independent of ϕ.

The various derivatives of this integral needed for the elastic field are evaluated in Appendix A. For the half space region $z > 0$, the above integral as well as its derivatives are continuous functions. From eqn (2) the displacements become

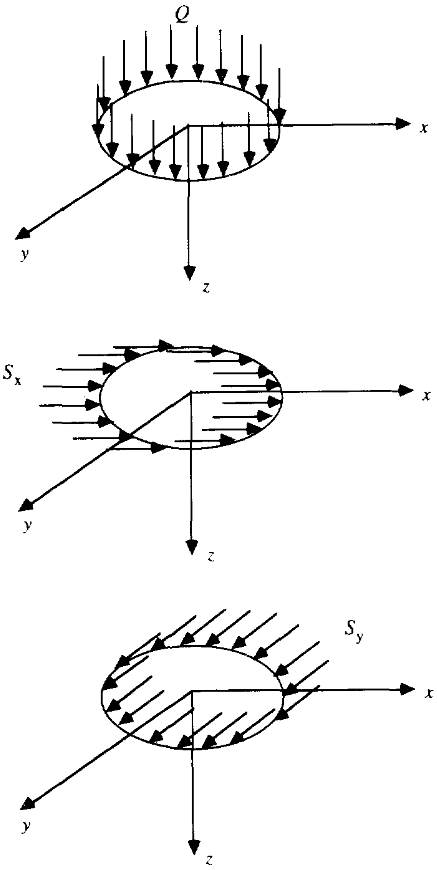

Fig. 2. Ring loadings in the x, y and z directions.

$$u' = \Lambda \sum_{j=1}^{2} F_j = HQa\,e^{i\phi} \sum_{j=1}^{2} \frac{\gamma_j}{(m_j-1)} \frac{\partial}{\partial \rho} \psi(\rho, z_j) = 2\pi HQa\,e^{i\phi} \sum_{j=1}^{2} \frac{\gamma_j}{(m_j-1)} I_j(0,1;0), \tag{36}$$

$$w = \sum_{j=1}^{2} m_j \frac{\partial}{\partial z} F_j = \sum_{j=1}^{2} \frac{m_j}{\gamma_j} \frac{\partial}{\partial z_j} F_j = 2\pi HQa \sum_{j=1}^{2} \frac{m_j}{(m_j-1)} I_j(0,0;0), \tag{37}$$

where the functions $I(\mu, v; \lambda)$ are given in Appendix D in terms of the complete elliptic integrals $\mathbf{F}(k)$, $\mathbf{E}(k)$ and $\Pi(n,k)$. The functions $I_j(\mu, v; \lambda)$ are obtained from $I(\mu, v; \lambda)$ by substituting $z \to z_j$ in these formulas. Thus $I_j(\mu, v; \lambda)$ are given in terms of $\mathbf{F}(k_j)$, $\mathbf{E}(k_j)$ and $\Pi(n_j, k_j)$.

The stress field can be evaluated in a similar manner leading to the results

$$\sigma_1 = -4\pi a HQA_{66} \sum_{j=1}^{2} \frac{\gamma_j^2 - (1+m_j)\gamma_3^2}{\gamma_j(m_j-1)} I_j(0,0;1), \tag{38}$$

$$\sigma_{zz} = -\frac{aQ}{(\gamma_1-\gamma_2)} \sum_{j=1}^{2} \gamma_j (-1)^{j+1} I_j(0,0;1), \tag{39}$$

$$\sigma_2 = 4\pi a HQA_{66}\, e^{i2\phi} \sum_{j=1}^{2} \frac{\gamma_j}{(m_j-1)} G_j(0,0;1), \tag{40}$$

$$\tau_z = -\frac{aQ\,e^{i\phi}}{(\gamma_1-\gamma_2)}\sum_{j=1}^{2}(-1)^{j+1}I_j(0,1;1), \tag{41}$$

where the combination $G(\mu,\nu;\lambda)$ is defined in Appendix D and the identity

$$\frac{(m_j+1)}{(m_j-1)} = \frac{(-1)^{j+1}}{2\pi HA_{44}(\gamma_1-\gamma_2)}, \tag{42}$$

has been used in eqns (39) and (41) above

5.2. *Ring shear loading in the* x *and* y *directions*

Attention is now focused on the case of uniform ring loading applied in the x and y directions as shown in Fig. 2. The force per unit circumferential length in the x and y directions are denoted as S_x and S_y, respectively. The potentials can be obtained from eqn (33) by setting $\rho_0 = a$ and replacing the complex force T with $Sa\,d\phi_0(S = S_x+iS_y)$ and integrating the result from $0 < \phi_0 < 2\pi$. The potentials are now

$$F_j(\rho,\phi,z) = \frac{aH\gamma_1\gamma_2}{2(m_j-1)}[S\bar{\Lambda}+\bar{S}\Lambda]\Gamma(\rho,z_j), \quad j=1,2,$$

$$F_3(\rho,\phi,z) = \frac{ai\gamma_3}{4\pi A_{44}}[S\bar{\Lambda}-\bar{S}\Lambda]\Gamma(\rho,z_3),$$

$$\Gamma(\rho,z) = \int_0^{2\pi}\chi(z)\,d\phi_0 = \int_0^{2\pi}[z\ln[R+z]-R]\,d\phi_0, \tag{43}$$

where R is defined in eqn (35). The derivatives of this integral are evaluated in Appendix B. Using results for the ρ derivative allows the potential functions to be written as

$$F_j(\rho,\phi,z) = -\frac{\pi aH\gamma_1\gamma_2}{(m_j-1)}[S\,e^{-i\phi}+\bar{S}e^{i\phi}]I_j(0,1;-1), \quad j=1,2,$$

$$F_3(\rho,\phi,z) = -\frac{ai\gamma_3}{2A_{44}}[S\,e^{-i\phi}-\bar{S}e^{i\phi}]I_3(0,1;-1). \tag{44}$$

The displacements and stresses for this case are

$$u^c = -\pi aH\gamma_1\gamma_2\sum_{j=1}^{2}\frac{1}{(m_j-1)}[SI_j(0,0;0)+\bar{S}e^{i2\phi}G_j(0,0;0)]$$

$$+\frac{a\gamma_3}{2A_{44}}[SI_3(0,0;0)-\bar{S}e^{i2\phi}G_3(0,0;0)], \tag{45}$$

$$w = 2\pi aH\gamma_1\gamma_2[S_x\cos\phi+S_y\sin\phi]\sum_{j=1}^{2}\frac{m_j}{\gamma_j(m_j-1)}I_j(0,1;0), \tag{46}$$

$$\sigma_1 = -4\pi aH\gamma_1\gamma_2A_{66}[S_x\cos\phi+S_y\sin\phi]\sum_{j=1}^{2}\frac{\gamma_j^2-(1+m_j)\gamma_3^2}{\gamma_j^2(m_j-1)}I_j(0,1;1), \tag{47}$$

$$\sigma_{zz} = -\frac{a\gamma_1\gamma_2}{(\gamma_1-\gamma_2)}[S_x\cos\phi+S_y\sin\phi]\sum_{j=1}^{2}(-1)^{j+1}I_j(0,1;1), \tag{48}$$

$$\sigma_z = 2\pi a H\gamma_1\gamma_2 A_{66}\sum_{j=1}^{2}\frac{1}{(m_j-1)}[S\,e^{i\phi}I_j(0,1;1)+\bar{S}\,e^{i3\phi}H_j(0,1;1)]$$

$$-\frac{a}{\gamma_3}[S\,e^{i\phi}I_3(0,1;1)-\bar{S}\,e^{i3\phi}H_3(0,1;1)], \tag{49}$$

$$\tau_z = \frac{a\gamma_1\gamma_2}{2(\gamma_1-\gamma_2)}\sum_{j=1}^{2}\frac{(-1)^{j+1}}{\gamma_j}[SI_j(0,0;1)+\bar{S}\,e^{i2\phi}G_j(0,0;1)]$$

$$-\frac{a}{2}[SI_3(0,0;1)-\bar{S}\,e^{i2\phi}G_3(0,0;1)], \tag{50}$$

where the combination $H(\mu,\nu;\lambda)$ is defined in Appendix D.

5.3. *Ring shear loading in the ρ and ϕ directions*

The final ring loadings investigated on the half space are the axisymmetric cases for shear loading in the radial and tangential directions shown in Fig. 3. The force per unit circumferential length in the ρ and ϕ directions are denoted as S_ρ and S_ϕ, respectively. The potentials can again be obtained from eqn (33) by setting $\rho_0 = a$ and replacing the complex force T with $Sa\,e^{i\phi_0}\,d\phi_0$ $(S = S_\rho + iS_\phi)$ and integrating the result from $0 < \phi_0 < 2\pi$. The potentials are

$$F_j(\rho,\phi,z) = \frac{aH\gamma_1\gamma_2}{2(m_j-1)}[S\bar{\Lambda}\Omega(\rho,\phi,z_j)+\bar{S}\Lambda\bar{\Omega}(\rho,\phi,z_j)], \quad j=1,2,$$

$$F_3(\rho,\phi,z) = \frac{ai\gamma_3}{4\pi A_{44}}[S\bar{\Lambda}\Omega(\rho,\phi,z_3)-\bar{S}\Lambda\bar{\Omega}(\rho,\phi,z_3)],$$

$$\Omega(\rho,\phi,z) = \int_0^{2\pi} e^{i\phi_0}\chi(z)\,d\phi_0, \quad \bar{\Omega}(\rho,\phi,z) = \int_0^{2\pi} e^{-i\phi_0}\chi(z)\,d\phi_0,$$

$$\chi(z) = z\ln[R+z]-R, \quad R^2 = \rho^2+a^2-2a\rho\cos(\phi-\phi_0)+z^2. \tag{51}$$

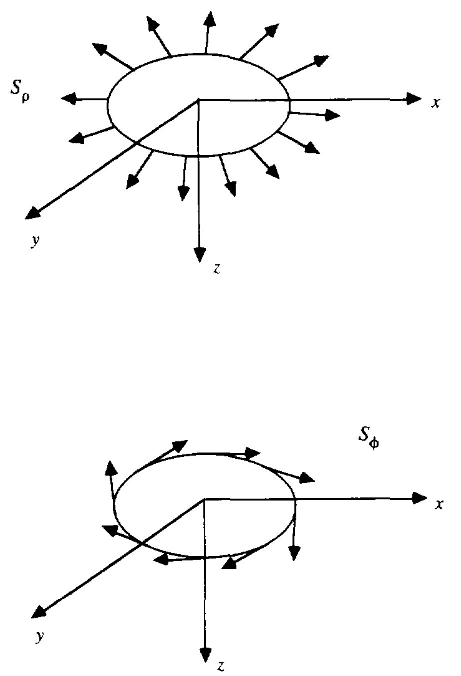

Fig. 3. Ring loadings in the ρ and ϕ directions.

The function $\Omega(\rho, \phi, z)$ and its derivatives are evaluated in Appendix C. The function $\Omega(\rho, \phi, z)$ is

$$\Omega(\rho, \phi, z) = 2\pi e^{i\phi} I(1, 1; -2), \tag{52}$$

from which the potentials become

$$F_j(\rho, \phi, z) = \frac{2\pi a H\gamma_1\gamma_2}{(m_j-1)} S_\rho \left[\frac{\partial}{\partial \rho} + \frac{1}{\rho}\right] I_j(1, 1; -2) = \frac{2\pi a H\gamma_1\gamma_2}{(m_j-1)} S_\rho I_j(1, 0; -1), \quad j = 1, 2,$$

$$F_3(\rho, \phi, z) = -\frac{a\gamma_3}{A_{44}} S_\phi \left[\frac{\partial}{\partial \rho} + \frac{1}{\rho}\right] I_3(1, 1; -2) = -\frac{a\gamma_3}{A_{44}} S_\phi I_3(1, 0; -1). \tag{53}$$

Differentiation of these potentials leads to the elastic field

$$u^c = -2\pi a H\gamma_1\gamma_2 S_\rho e^{i\phi} \sum_{j=1}^{2} \frac{1}{(m_j-1)} I_j(1, 1; 0) + \frac{ai\gamma_3}{A_{44}} S_\phi e^{i\phi} I_3(1, 1; 0), \tag{54}$$

$$w = -2\pi a H\gamma_1\gamma_2 S_\rho \sum_{j=1}^{2} \frac{m_j}{\gamma_j(m_j-1)} I_j(1, 0; 0), \tag{55}$$

$$\sigma_1 = 4\pi a H\gamma_1\gamma_2 A_{66} S_\rho \sum_{j=1}^{2} \frac{\gamma_j^2 - (1+m_j)\gamma_3^2}{\gamma_j^2(m_j-1)} I_j(1, 0; 1), \tag{56}$$

$$\sigma_{zz} = \frac{a\gamma_1\gamma_2}{(\gamma_1-\gamma_2)} S_\rho \sum_{j=1}^{2} (-1)^{j+1} I_j(1, 0; 1), \tag{57}$$

$$\sigma_2 = -4\pi a H\gamma_1\gamma_2 A_{66} S_\rho e^{i2\phi} \sum_{j=1}^{2} \frac{1}{(m_j-1)} G_j(1, 0; 1) + \frac{2ia}{\gamma_3} S_\phi e^{i2\phi} G_3(1, 0; 1), \tag{58}$$

$$\tau_z = \frac{a\gamma_1\gamma_2}{(\gamma_1-\gamma_2)} S_\rho e^{i\phi} \sum_{j=1}^{2} \frac{(-1)^{j+1}}{\gamma_j} I_j(1, 1; 1) - iaS_\phi e^{i\phi} I_3(1, 1; 1). \tag{59}$$

The corresponding results for an isotropic half space can be found in Appendix G for the three cases of ring loading considered above.

The displacement fields for the loadings Q, S_ρ and S_ϕ have been previously given by Hasegawa and Watanabe (1995). Their results for Q and S_ϕ are in agreement with the present displacements if one substitutes $-1/(2\pi a)$ for Q and S_ϕ and the relations in eqn (H29) are used. The same procedure does not lead to an agreement for the displacements caused by S_ρ. It appears that the expressions in eqns (9) and (12) of their paper may contain misprints.

6. RING LOADING IN A FULL SPACE

Consider now the transversely isotropic full space where the $z = 0$ plane is an isotropic plane. Again using cylindrical coordinates, a point force is applied on the $z = 0$ plane at ρ_0, ϕ_0 with components T_x, T_y and P in the x, y and z directions. The potential functions for these fundamental point force solutions were also given by Fabrikant (1989). For the point normal force P, the potentials are

$$F_j(\rho,\phi,z;\rho_0,\phi_0) = \frac{(-1)^{j+1}P}{4\pi A_{44}(m_1-m_2)}\ln[R_j+z_j],\quad j=1,2,$$

$$F_3(\rho,\phi,z;\rho_0,\phi_0) = 0. \tag{60}$$

with R_j defined in eqn (31). For the shear forces the potentials are

$$F_j(\rho,\phi,z;\rho_0,\phi_0) = \frac{(-1)^{j+1}\gamma_j}{8\pi A_{44}(m_1-m_2)m_j}(T\bar{\Lambda}+\bar{T}\Lambda)\chi(z_j),\quad j=1,2$$

$$F_3(\rho,\phi,z;\rho_0,\phi_0) = \frac{i\gamma_3}{8\pi A_{44}}(T\bar{\Lambda}-\bar{T}\Lambda)\chi(z_3), \tag{61}$$

where $\chi(z_j)$ is also as defined above. Notice that the only difference in the potentials here, as compared to the half space, are the constants.

6.1. *Ring normal loading*

Consider a ring normal load on the $z=0$ plane located at a radius of $\rho=a$ with a density of Q. Following the above procedure the potentials become

$$F_j(\rho,\phi,z) = \frac{Qa(-1)^{j-1}}{4\pi A_{44}(m_1-m_2)}\psi(\rho,z_j),\quad j=1,2. \tag{62}$$

where the derivatives of $\psi(\rho,z_j)$ are found in Appendix A. Note that for $z>0$, the derivatives of $\psi(\rho,z_j)$ are continuous functions but for $z<0$ the radial derivatives of $\psi(\rho,z_j)$ are discontinuous. To see the effect of this discontinuity, consider the complex displacement u^c which is given as

$$u^c = \Lambda\sum_{j=1}^{2}F_j = \frac{Qa\,e^{i\phi}}{4\pi A_{44}(m_1-m_2)}\sum_{j=1}^{2}(-1)^{j+1}\frac{\partial}{\partial\rho}\psi(\rho,z_j)$$

$$= \frac{Qa\,e^{i\phi}}{4\pi A_{44}(m_1-m_2)}\sum_{j=1}^{2}(-1)^{j+1}\left[2\pi I_j(0,1;0)+\left\{0, z>0;\ -\frac{2\pi}{\rho}[1-\mathrm{sgn}(\rho-a)],\quad z<0\right\}\right]$$

$$= \frac{Qa\,e^{i\phi}}{2A_{44}(m_1-m_2)}\sum_{j=1}^{2}(-1)^{j-1}I_j(0,1;0). \tag{63}$$

Since the discontinuous term is independent of z_j it cancels in the summation process. Thus the displacements are continuous functions as they should be. Also note that the $z=0$ plane is one of anti-symmetry and thus u^c should be an odd function of z. The first term of $I_j(0,1;0)$ is ρ^{-1} which is an even function while the second term of $I_j(0,1;0)$ is an odd function. The first term cancels in the summation, again yielding the correct solution behavior. The displacement w is

$$w = \sum_{j=1}^{2}m_j\frac{\partial}{\partial z}F_j = \sum_{j=1}^{2}\frac{m_j}{\gamma_j}\frac{\partial}{\partial z_j}F_j = \frac{Qa}{2A_{44}(m_1-m_2)}\sum_{j=1}^{2}\frac{m_j(-1)^{j+1}}{\gamma_j}I_j(0,0;0), \tag{64}$$

which is an even function of z.

To evaluate the stress components, σ_1 and σ_{zz} are considered first. They can be evaluated as

$$\sigma_1 = -\frac{aQA_{66}}{A_{44}(m_1-m_2)}\sum_{j=1}^{2}\frac{(-1)^{j+1}[\gamma_j^2-(1+m_j)\gamma_3^2]}{\gamma_j^2}I_j(0,0;1), \tag{65}$$

$$\sigma_{zz} = -\frac{aQ}{2(m_1-m_2)}\sum_{j=1}^{2}(-1)^{j+1}(1+m_j)I_j(0,0;1). \tag{66}$$

The stress component σ_2 is now determined. It has the form

$$\sigma_2 = \frac{aQA_{66}\,e^{i2\phi}}{2\pi A_{44}(m_1-m_2)}\sum_{j=1}^{2}(-1)^{j+1}\left\{\frac{\partial^2}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right\}\psi(\rho,z_j)$$

$$= \frac{aQA_{66}\,e^{i2\phi}}{A_{44}(m_1-m_2)}\sum_{j=1}^{2}(-1)^{j+1}G_j(0,0;1), \tag{67}$$

where again the discontinuous terms sum to zero. The last stress component is τ_z which becomes

$$\tau_z = -\frac{aQ\,e^{i\phi}}{2(m_1-m_2)}\sum_{j=1}^{2}\frac{(-1)^{j+1}(1+m_j)}{\gamma_j}I_j(0,1;1). \tag{68}$$

The above elastic field has the same functional form in the region $z>0$ and $z<0$ since the discontinuities cancel. This solution is identical to the corresponding half space result except for the differing constants multiplying each term.

6.2. *Ring shear loading in the* x *and* y *directions*

The elastic field for uniform ring loading applied in the x and y directions will now be found. Using S_x and S_y as before, the potentials are now

$$F_j(\rho,\phi,z) = \frac{a\gamma_j(-1)^{j+1}}{8\pi A_{44}(m_1-m_2)m_j}[S\bar{\Lambda}+\bar{S}\Lambda]\Gamma(\rho,z_j), \quad j=1,2,$$

$$F_3(\rho,\phi,z) = \frac{ai\gamma_3}{8\pi A_{44}}[S\bar{\Lambda}-\bar{S}\Lambda]\Gamma(\rho,z_3), \tag{69}$$

where $\Gamma(\rho,z_j)$ is defined in eqn (43) and its derivatives are evaluated in Appendix B. Using equation (B5) these potentials are

$$F_j(\rho,\phi,z) = \frac{a\gamma_j(-1)^{j+1}}{8\pi A_{44}(m_1-m_2)m_j}[S\,e^{-i\phi}+\bar{S}\,e^{i\phi}]\Bigg[-2\pi I_j(0,1;-1)$$

$$+\left\{0, z>0;\ -\frac{2\pi z_j}{\rho}[1-\mathrm{sgn}(\rho-a)], z<0\right\}\Bigg], \quad j=1,2,$$

$$F_3(\rho,\phi,z) = \frac{ai\gamma_3}{8\pi A_{44}}[S\,e^{-i\phi}-\bar{S}\,e^{i\phi}]\Bigg[-2\pi I_3(0,1;-1)$$

$$+\left\{0, z>0;\ -\frac{2\pi z_3}{\rho}[1-\mathrm{sgn}(\rho-a)], z<0\right\}\Bigg]. \tag{70}$$

This displays that the potential functions can be explicitly evaluated in terms of discontinuous functions however it is more convenient to substitute eqns (69) into the expressions for the elastic field.

Proceeding in this manner, the displacements are now found. The in-plane displacements become

$$u^c = \frac{a}{8\pi A_{44}(m_1-m_2)}\sum_{j=1}^{2}\frac{\gamma_j(-1)^{j+1}}{m_j}[S\Delta+\bar{S}\Lambda^2]\Gamma(\rho,z_j)-\frac{a\gamma_3}{8\pi A_{44}}[S\Delta-\bar{S}\Lambda^2]\Gamma(\rho,z_3). \tag{71}$$

Here it is noted that $\Delta\Gamma(\rho,z_j)$ is a continuous function but $\Lambda^2\Gamma(\rho,z_j)$ is not. It is also apparent that the constant inside the summation sign is also more complicated than in the previous loading example. Nevertheless it will now be shown that the discontinuous terms cancel. Taking only the discontinuous terms for $z<0$ leads to the expression

$$\begin{aligned} u^c &= \frac{a}{8\pi A_{44}(m_1-m_2)}\sum_{j=1}^{2}\frac{\gamma_j(-1)^{j+1}}{m_j}\bar{S}e^{i2\phi}\frac{2\pi z_j}{\rho^2}[1-\operatorname{sgn}(\rho-a)] \\ &\quad+\frac{a\gamma_3}{8\pi A_{44}}\bar{S}e^{i2\phi}\frac{2\pi z_3}{\rho^2}[1-\operatorname{sgn}(\rho-a)], \\ &= \frac{a}{8\pi A_{44}(m_1-m_2)}\bar{S}e^{i2\phi}\frac{2\pi z}{\rho^2}[1-\operatorname{sgn}(\rho-a)]\sum_{j=1}^{2}\frac{(-1)^{j+1}}{m_j} \\ &\quad+\frac{a}{8\pi A_{44}}\bar{S}e^{i2\phi}\frac{2\pi z}{\rho^2}[1-\operatorname{sgn}(\rho-a)], \end{aligned} \tag{72}$$

where the relation $\gamma_j z_j = z$ has been used. The sum can be evaluated as

$$\sum_{j=1}^{2}\frac{(-1)^{j+1}}{m_j}=\frac{1}{m_1}-\frac{1}{m_2}=\frac{m_2-m_1}{m_1m_2}=m_2-m_1, \tag{73}$$

since $m_1m_2=1$ (Fabrikant, 1989). It is seen that the discontinuity in the two term summation resulting from F_1 and F_2 cancels with the one caused by F_3. Taking the remaining terms provides the final result

$$\begin{aligned} u^c &= -\frac{a}{4A_{44}(m_1-m_2)}\sum_{j=1}^{2}\frac{\gamma_j(-1)^{j+1}}{m_j}[SI_j(0,0;0)+\bar{S}e^{i2\phi}G_j(0,0;0)] \\ &\quad+\frac{a\gamma_3}{4A_{44}}[SI_3(0,0;0)-\bar{S}e^{i2\phi}G_3(0,0;0)]. \end{aligned} \tag{74}$$

In this case u^c should be an even function of z however the first term of $I_j(0,1;-1)$ is an odd function. This first term cancels in the summation process analogous to the discontinuous term above. The z directed displacement can be found as

$$\begin{aligned} w &= \sum_{j=1}^{2} m_j\frac{\partial}{\partial z}F_j=\sum_{j=1}^{2}\frac{m_j}{\gamma_j}\frac{\partial}{\partial z_j}F_j=\frac{a}{8\pi A_{44}(m_1-m_2)}\sum_{j=1}^{2}(-1)^{j+1}[Se^{-i\phi}+\bar{S}e^{i\phi}]\frac{\partial}{\partial\rho}\psi(\rho,z_j) \\ &= \frac{a}{8\pi A_{44}(m_1-m_2)}[Se^{-i\phi}+\bar{S}e^{i\phi}]\sum_{j=1}^{2}(-1)^{j+1}\Bigg[2\pi I_j(0,1;0) \\ &\quad+\left\{0,z>0;-\frac{2\pi}{\rho}[1-\operatorname{sgn}(\rho-a)],z<0\right\}\Bigg]. \end{aligned} \tag{75}$$

It is easy to see that the discontinuous term cancels leading to

$$w = \frac{a}{2A_{44}(m_1-m_2)}[S_x \cos\phi + S_y \sin\phi] \sum_{j=1}^{2} (-1)^{j+1} I_j(0,1;0). \tag{76}$$

The stresses are now derived. The components σ_1 and σ_{zz} do not have discontinuities to consider and they can be easily evaluated as

$$\sigma_1 = -\frac{aA_{66}}{A_{44}(m_1-m_2)}[S_x \cos\phi + S_y \sin\phi] \sum_{j=1}^{2} \frac{(-1)^{j+1}[\gamma_j^2-(1+m_j)\gamma_3^2]}{\gamma_j m_j} I_j(0,1;1), \tag{77}$$

$$\sigma_{zz} = -\frac{a}{2(m_1-m_2)}[S_x \cos\phi + S_y \sin\phi] \sum_{j=1}^{2} \frac{(-1)^{j+1}\gamma_j(1+m_j)}{m_j} I_j(0,1;1). \tag{78}$$

Now σ_2 is written as

$$\sigma_2 = \frac{aA_{66}}{4\pi A_{44}(m_1-m_2)} \sum_{j=1}^{2} \frac{\gamma_j(-1)^{j+1}}{m_j}[S\Delta\Lambda + \bar{S}\Lambda^3]\Gamma(\rho, z_j) - \frac{aA_{66}\gamma_3}{4\pi A_{44}}[S\Delta\Lambda - \bar{S}\Lambda^3]\Gamma(\rho, z_3). \tag{79}$$

The terms involving $\Lambda^3\Gamma(\rho, z_j)$ are again discontinuous but cancel in the same manner as for u^c above. The resulting expression has the form

$$\sigma_2 = \frac{aA_{66}}{2A_{44}(m_1-m_2)} \sum_{j=1}^{2} \frac{\gamma_j(-1)^{j+1}}{m_j}[S e^{i\phi} I_j(0,1;1) + \bar{S} e^{i3\phi} H_j(0,1;1)]$$
$$- \frac{a}{2\gamma_3}[S e^{i\phi} I_3(0,1;1) - \bar{S} e^{i3\phi} H_3(0,1;1)]. \tag{80}$$

The last stress component is

$$\tau_z = \frac{a}{8\pi(m_1-m_2)} \sum_{j=1}^{2} \frac{(-1)^{j+1}(1+m_j)}{m_j}[S\Delta + \bar{S}\Lambda^2]\psi(\rho, z_j) - \frac{a}{8\pi}[S\Delta - \bar{S}\Lambda^2]\psi(\rho, z_3). \tag{81}$$

The function $\Lambda^2\psi(\rho, z_j)$ is discontinuous. Considering these terms lead to

$$\tau_z = \frac{a}{8\pi(m_1-m_2)} \sum_{j=1}^{2} \frac{(-1)^{j+1}(1+m_j)}{m_j} \bar{S} e^{i2\phi} \left\{0, z>0 ; \frac{4\pi}{\rho^2}[1-\mathrm{sgn}(\rho-a)], z<0\right\}$$
$$+ \frac{a}{8\pi} \bar{S} e^{i2\phi} \left\{0, z>0 ; \frac{4\pi}{\rho^2}[1-\mathrm{sgn}(\rho-a)], z<0\right\}. \tag{82}$$

Using the result

$$\sum_{j=1}^{2} \frac{(-1)^{j+1}(1+m_j)}{m_j} = \frac{(1+m_1)}{m_1} - \frac{(1+m_2)}{m_2} = \frac{(m_2-m_1)}{m_1 m_2} = m_2 - m_1, \tag{83}$$

illustrates that these terms cancel leaving

$$\tau_z = \frac{a}{4(m_1-m_2)}\sum_{j=1}^{2}\frac{(-1)^{j+1}(1+m_j)}{m_j}[SI_j(0,0;1)+\bar{S}e^{i2\phi}G_j(0,0;1)]$$

$$-\frac{a}{4}[SI_3(0,0;1)-\bar{S}e^{i2\phi}G_3(0,0;1)]. \quad (84)$$

6.3. *Ring shear loading in the ρ and ϕ directions*

The axisymmetric shear loading solutions are now found. Proceeding as in Section 5.3 while using the full space potentials in eqn (61) gives the potentials as

$$F_j(\rho,\phi,z) = \frac{a\gamma_j(-1)^{j+1}}{8\pi A_{44}(m_1-m_2)m_j}[S\bar{\Lambda}\Omega(\rho,\phi,z_j)+\bar{S}\Lambda\bar{\Omega}(\rho,\phi,z_j)],\quad j=1,2,$$

$$F_3(\rho,\phi,z) = \frac{ai\gamma_3}{8\pi A_{44}}[S\bar{\Lambda}\Omega(\rho,\phi,z_3)-\bar{S}\Lambda\bar{\Omega}(\rho,\phi,z_3)], \quad (85)$$

with $\Omega(\rho,\phi,z_j)$ and $\bar{\Omega}(\rho,\phi,z_j)$ defined in eqn (51) and evaluated in Appendix C along with some useful derivatives. From eqns (C9) and (C10) these potentials can be rewritten as

$$F_j(\rho,\phi,z) = \frac{a\gamma_j(-1)^{j+1}}{4\pi A_{44}(m_1-m_2)m_j}S_\rho\left\{\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right\}f(\rho,z_j) = \frac{a\gamma_j(-1)^{j+1}}{2A_{44}(m_1-m_2)m_j}S_\rho\Bigg[I_j(1,0;-1)$$

$$+\left\{0,z>0;-\frac{z_j}{a}[1-\mathrm{sgn}(\rho-a)],z<0\right\}\Bigg],\quad j=1,2,$$

$$F_3(\rho,\phi,z) = -\frac{a\gamma_3}{4\pi A_{44}}S_\phi\left\{\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right\}f(\rho,z_3) = -\frac{a\gamma_3}{2A_{44}}S_\phi\Bigg[I_3(1,0;-1)$$

$$+\left\{0,z>0;-\frac{z_3}{a}[1-\mathrm{sgn}(\rho-a)],z<0\right\}\Bigg]. \quad (86)$$

The elastic field can be found by differentiating the above potentials. All discontinuous terms cancel and the final results can be given as

$$u^c = -\frac{a}{2A_{44}(m_1-m_2)}S_\rho e^{i\phi}\sum_{j=1}^{2}\frac{\gamma_j(-1)^{j+1}}{m_j}I_j(1,1;0)+\frac{ia\gamma_3}{2A_{44}}S_\phi e^{i\phi}I_3(1,1;0), \quad (87)$$

$$w = -\frac{a}{2A_{44}(m_1-m_2)}S_\rho\sum_{j=1}^{2}(-1)^{j+1}I_j(1,0;0). \quad (88)$$

$$\sigma_1 = \frac{aA_{66}}{A_{44}(m_1-m_2)}S_\rho\sum_{j=1}^{2}\frac{(-1)^{j+1}[\gamma_j^2-(1+m_j)\gamma_3^2]}{\gamma_j m_j}I_j(1,0;1), \quad (89)$$

$$\sigma_{zz} = \frac{a}{2(m_1-m_2)}S_\rho\sum_{j=1}^{2}\frac{(-1)^{j+1}(1+m_j)\gamma_j}{m_j}I_j(1,0;1), \quad (90)$$

$$\sigma_2 = -\frac{aA_{66}}{A_{44}(m_1-m_2)}S_\rho e^{i2\phi}\sum_{j=1}^{2}\frac{\gamma_j(-1)^{j+1}}{m_j}G_j(1,0;1)+\frac{ai}{\gamma_3}S_\phi e^{i2\phi}G_3(1,0;1), \quad (91)$$

$$\tau_z = \frac{a}{2(m_1-m_2)}S_\rho e^{i\phi}\sum_{j=1}^{2}\frac{(-1)^{j+1}(1+m_j)}{m_j}I_j(1,1;1)-\frac{ai}{2}S_\phi e^{i\phi}I_3(1,1;1). \quad (92)$$

The corresponding results for an isotropic full space can be found in Appendix H for the three loading cases considered above.

7. RING LOADING BURIED IN A HALF SPACE

The last loading case which will be examined is when the ring loads are applied below the surface of a half space $z > 0$. The potentials for a buried point force perpendicular to the surface were first found by Shield (1951). It is derived in Appendix E using the present notation. For a force of magnitude P in the z direction at the point (ρ_0, ϕ_0, h) the potentials are

$$F_j(\rho, \phi, z; \rho_0, \phi_0) = \frac{(-1)^{j+1}P}{4\pi A_{44}(m_1-m_2)}\left[\ln[R'_j+z'_j]-\ln[R''_j+z''_j] + \frac{2\gamma_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}(-1)^{n+1}(m_n+1)\ln[R_{jn}+(z_j+h_n)]\right], \quad j=1,2,$$

$$h_n = \frac{h}{\gamma_n}, \quad z'_j = z_j - h_j, \quad z''_j = z_j + h_j, \quad R'^2_j = \rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z'^2_j,$$

$$R''^2_j = \rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z''^2_j, \quad R^2_{jn} = \rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+(z_j+h_n)^2. \tag{93}$$

The potentials for a buried force parallel to the surface are also derived in Appendix E. For a force with magnitudes T_x and T_y at the point (ρ_0, ϕ_0, h) the potentials are

$$F_j(\rho, \phi, z; \rho_0, \phi_0) = \frac{(-1)^{j+1}\gamma_j}{8\pi A_{44}(m_1-m_2)}\left[\frac{1}{m_j}(T\bar{\Lambda}+\bar{T}\Lambda)\{\chi(z'_j)+\chi(z''_j)\} - \frac{2}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n+1}(m_n+1)\gamma_n}{m_n}(T\bar{\Lambda}+\bar{T}\Lambda)\chi(z_j+h_n)\right], \quad j=1,2,$$

$$F_3(\rho, \phi, z; \rho_0, \phi_0) = \frac{i\gamma_3}{8\pi A_{44}}(T\bar{\Lambda}-\bar{T}\Lambda)\{\chi(z'_3)+\chi(z''_3)\},$$

$$\chi(z'_j) = z'_j\ln[R'_j+z'_j]-R'_j, \quad \chi(z''_j) = z''_j\ln[R''_j+z''_j]-R''_j, \tag{94}$$

$$\chi(z_j+h_n) = (z_j+h_n)\ln[R_{jn}+(z_j+h_n)]-R_{jn}.$$

7.1. *Ring normal loading*

Integrating eqn (93) leads to the potentials as

$$F_j(\rho, \phi, z) = \frac{(-1)^{j+1}Qa}{4\pi A_{44}(m_1-m_2)}\left[\psi(\rho, z'_j)-\psi(\rho, z''_j) + \frac{2\gamma_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}(-1)^{n-1}(m_n+1)\psi(\rho, z_j+h_n)\right], \quad j=1,2, \tag{95}$$

where $\psi(\rho, z)$ is defined in eqn (35). To evaluate the elastic field the following notation is adopted

$$I'_j(\mu, v; \lambda) = I(\mu, v; \lambda), \quad z\to z'_j; \quad I''_j(\mu, v; \lambda) = I(\mu, v; \lambda), \quad z\to z''_j$$

$$I_{jn}(\mu, v; \lambda) = I(\mu, v; \lambda), \quad z\to z_j+h_n. \tag{96}$$

The first term for the potentials above corresponds to a point force in a full space and its radial derivatives will be discontinuous in the region $0 < z < h$ since z_j' has a negative real part. However these will cancel in the summation process as above. For the image force and half space correction terms, z_j'' and z_j+h_n always have a positive real part and no discontinuities occur. Using the derivatives in Appendix A the elastic field is

$$u^c = \frac{Qa\,e^{i\phi}}{2A_{44}(m_1-m_2)}\sum_{j=1}^{2}(-1)^{j+1}\Bigg[I_j'(0,1;0)-I_j''(0,1;0) + \frac{2\gamma_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}(-1)^{n+1}(m_n+1)I_{jn}(0,1;0)\Bigg], \tag{97}$$

$$w = \frac{Qa}{2A_{44}(m_1-m_2)}\sum_{j=1}^{2}\frac{(-1)^{j+1}m_j}{\gamma_j}\Bigg[I_j'(0,0;0)-I_j''(0,0;0) + \frac{2\gamma_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}(-1)^{n+1}(m_n+1)I_{jn}(0,0;0)\Bigg], \tag{98}$$

$$\sigma_1 = \frac{-aQA_{66}}{A_{44}(m_1-m_2)}\sum_{j=1}^{2}\frac{(-1)^{j+1}[\gamma_j^2-(1+m_j)\gamma_3^2]}{\gamma_j^2}\Bigg[I_j'(0,0;1)-I_j''(0,0;1) + \frac{2\gamma_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}(-1)^{n+1}(m_n+1)I_{jn}(0,0;1)\Bigg], \tag{99}$$

$$\sigma_{zz} = \frac{-aQ}{2(m_1-m_2)}\sum_{j=1}^{2}(-1)^{j-1}(1+m_j)\Bigg[I_j'(0,0;1)-I_j''(0,0;1) + \frac{2\gamma_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}(-1)^{n-1}(m_n+1)I_{jn}(0,0;1)\Bigg], \tag{100}$$

$$\sigma_2 = \frac{QaA_{66}\,e^{i2\phi}}{A_{44}(m_1-m_2)}\sum_{j=1}^{2}(-1)^{j+1}\Bigg[G_j'(0,0;1)-G_j''(0,0;1) + \frac{2\gamma_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}(-1)^{n-1}(m_n+1)G_{jn}(0,0;1)\Bigg]. \tag{101}$$

$$\tau_z = \frac{-aQ}{2(m_1-m_2)}e^{i\phi}\sum_{j=1}^{2}\frac{(-1)^{j+1}(1+m_j)}{\gamma_j}\Bigg[I_j'(0,1;1)-I_j''(0,1;1) + \frac{2\gamma_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}(-1)^{n+1}(m_n+1)I_{jn}(0,1;1)\Bigg]. \tag{102}$$

7.2. *Ring shear loading in the* x *and* y *directions*

Integrating eqn (94) gives the potentials as

$$F_j(\rho,\phi,z) = \frac{(-1)^{j+1}\gamma_j a}{8\pi A_{44}(m_1-m_2)m_j}\Bigg[(S\bar{\Lambda}+\bar{S}\Lambda)\{\Gamma(\rho,z_j')+\Gamma(\rho,z_j'')\} - \frac{2m_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n+1}(m_n+1)\gamma_n}{m_n}(S\bar{\Lambda}+\bar{S}\Lambda)\Gamma(\rho,z_j+h_n)\Bigg],\quad j=1,2,$$

$$F_3(\rho,\phi,z) = \frac{ia\gamma_3}{8\pi A_{44}}(S\bar{\Lambda}-\bar{S}\Lambda)\{\Gamma(\rho,z_3')+\Gamma(\rho,z_3'')\}, \tag{103}$$

with $\Gamma(\rho, z)$ defined in eqn (43). Using the derivatives in Appendix B provides the elastic field as

$$\begin{aligned} u^c = \frac{-a}{4A_{44}(m_1-m_2)}\sum_{j=1}^{2}\frac{(-1)^{j-1}\gamma_j}{m_j}\Bigg[& S[I_j'(0,0;0)+I_j''(0,0;0)] \\ & +\bar{S}e^{i2\phi}[G_j'(0,0;0)+G_j''(0,0;0)]-\frac{2m_j}{(m_j+1)(\gamma_1-\gamma_2)} \\ & \times\sum_{n=1}^{2}\frac{(-1)^{n+1}(m_n+1)\gamma_n}{m_n}\{SI_{jn}(0,0;0)+\bar{S}e^{i2\phi}G_{jn}(0,0;0)\}\Bigg] \\ & +\frac{a\gamma_3}{4A_{44}}\Big[S[I_3'(0,0;0)+I_3''(0,0;0)]-\bar{S}e^{i2\phi}[G_3'(0,0;0)+G_3''(0,0;0)]\Big], \end{aligned} \tag{104}$$

$$\begin{aligned} w = \frac{a}{2A_{44}(m_1-m_2)}[S_x\cos\phi+S_y\sin\phi]\sum_{j=1}^{2}(-1)^{j+1}\Bigg[& I_j'(0,1;0)+I_j''(0,1;0) \\ & -\frac{2m_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n+1}(m_n+1)\gamma_n}{m_n}I_{jn}(0,1;0)\Bigg], \end{aligned} \tag{105}$$

$$\begin{aligned} \sigma_1 = & -\frac{aA_{66}}{A_{44}(m_1-m_2)}[S_x\cos\phi+S_y\sin\phi]\sum_{j=1}^{2}\frac{(-1)^{j+1}[\gamma_j^2-(1+m_j)\gamma_3^2]}{\gamma_j m_j} \\ & \times\Bigg[I_j'(0,1;1)+I_j''(0,1;1)-\frac{2m_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n-1}(m_n+1)\gamma_n}{m_n}I_{jn}(0,1;1)\Bigg], \end{aligned} \tag{106}$$

$$\begin{aligned} \sigma_{zz} = -\frac{a}{2(m_1-m_2)}[S_x\cos\phi+S_y\sin\phi]\sum_{j=1}^{2}\frac{(-1)^{j+1}(m_j+1)\gamma_j}{m_j}\Bigg[& I_j'(0,1;1)+I_j''(0,1;1) \\ & -\frac{2m_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n+1}(m_n+1)\gamma_n}{m_n}I_{jn}(0,1;1)\Bigg], \end{aligned} \tag{107}$$

$$\begin{aligned} \sigma_2 = \frac{aA_{66}}{2A_{44}(m_1-m_2)}\sum_{j=1}^{2}\frac{(-1)^{j+1}\gamma_j}{m_j}\Bigg[& Se^{i\phi}[I_j'(0,1;1)+I_j''(0,1;1)]+\bar{S}e^{i3\phi}[H_j'(0,1;1) \\ & +H_j''(0,1;1)]-\frac{2m_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n+1}(m_n+1)\gamma_n}{m_n}\{Se^{i\phi}I_{jn}(0,1;1) \\ & +\bar{S}e^{i3\phi}H_{jn}(0,1;1)\}\Bigg]-\frac{a}{2\gamma_3}[Se^{i\phi}[I_3'(0,1;1)+I_3''(0,1;1)]-\bar{S}e^{i3\phi}[H_3'(0,1;1) \\ & +H_3''(0,1;1)]], \end{aligned} \tag{108}$$

$$\begin{aligned} \tau_z = \frac{a}{4(m_1-m_2)}\sum_{j=1}^{2}\frac{(-1)^{j+1}(1+m_j)}{m_j}\Bigg[& S[I_j'(0,0;1)+I_j''(0,0;1)]+\bar{S}e^{i2\phi}[G_j'(0,0;1) \\ & +G_j''(0,0;1)]-\frac{2m_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n+1}(m_n+1)\gamma_n}{m_n}\{SI_{jn}(0,0;1) \end{aligned}$$

$$+\bar{S}e^{i2\phi}G_{jn}(0,0;1)\}\Bigg]-\frac{a}{4}[S[I_3'(0,0;1)+I_3''(0,0;1)]-\bar{S}e^{i2\phi}[G_3'(0,0;1)$$

$$+G_3''(0,0;1)]]. \tag{109}$$

7.3. *Ring shear loading in the ρ and ϕ directions*

Integrating eqn (94) gives the potentials as

$$F_j(\rho,\phi,z)=\frac{(-1)^{j+1}\gamma_j a}{4\pi A_{44}(m_1-m_2)m_j}S_\rho\Bigg[\left\{\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right\}\{f(\rho,z_j')+f(\rho,z_j'')\}$$

$$-\frac{2m_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n+1}(m_n+1)\gamma_n}{m_n}\left\{\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right\}f(\rho,z_j+h_n)\Bigg],\quad j=1,2,$$

$$F_3(\rho,\phi,z)=-\frac{a\gamma_3}{4\pi A_{44}}S_\phi\left\{\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right\}\{f(\rho,z_3')+f(\rho,z_3'')\}, \tag{110}$$

where eqn (C10) can be used to evaluate the above in terms of $I(1,0;-1)$ and a discontinuous term. Using the differential relations in Appendix C allows the elastic field to be found as

$$u^c=\frac{-a}{2A_{44}(m_1-m_2)}S_\rho e^{i\phi}\sum_{j=1}^{2}\frac{(-1)^{j+1}\gamma_j}{m_j}\Bigg[I_j'(1,1;0)+I_j''(1,1;0)$$

$$-\frac{2m_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n+1}(m_n+1)\gamma_n}{m_n}I_{jn}(1,1;0)\Bigg]$$

$$+\frac{ia\gamma_3}{2A_{44}}S_\phi e^{i\phi}\{I_3'(1,1;0)+I_3''(1,1;0)\}, \tag{111}$$

$$w=-\frac{a}{2A_{44}(m_1-m_2)}S_\rho\sum_{j=1}^{2}(-1)^{j-1}\Bigg[I_j'(1,0;0)+I_j''(1,0;0)$$

$$-\frac{2m_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n+1}(m_n+1)\gamma_n}{m_n}I_{jn}(1,0;0)\Bigg], \tag{112}$$

$$\sigma_1=\frac{aA_{66}}{A_{44}(m_1-m_2)}S_\rho\sum_{j=1}^{2}\frac{(-1)^{j+1}[\gamma_j^2-(1+m_j)\gamma_3^2]}{\gamma_j m_j}\Bigg[I_j'(1,0;1)+I_j''(1,0;1)$$

$$-\frac{2m_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n+1}(m_n+1)\gamma_n}{m_n}I_{jn}(1,0;1)\Bigg], \tag{113}$$

$$\sigma_{zz}=\frac{a}{2(m_1-m_2)}S_\rho\sum_{j=1}^{2}\frac{(-1)^{j+1}(m_j+1)\gamma_j}{m_j}\Bigg[I_j'(1,0;1)+I_j''(1,0;1)$$

$$-\frac{2m_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n+1}(m_n+1)\gamma_n}{m_n}I_{jn}(1,0;1)\Bigg], \tag{114}$$

$$\sigma_2=-\frac{aA_{66}}{A_{44}(m_1-m_2)}S_\rho e^{i2\phi}\sum_{j=1}^{2}\frac{(-1)^{j+1}\gamma_j}{m_j}\Bigg[G_j'(1,0;1)+G_j''(1,0;1)$$

$$-\frac{2m_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n+1}(m_n+1)\gamma_n}{m_n}G_{jn}(1,0;1)\Bigg]$$

$$+\frac{ia}{\gamma_3}S_\phi e^{i2\phi}[G_3'(1,0;1)+G_3''(1,0;1)], \quad (115)$$

$$\tau_z=\frac{a}{2(m_1-m_2)}S_\rho e^{i\phi}\sum_{j=1}^{2}\frac{(-1)^{j+1}(1+m_j)}{m_j}\Bigg[I_j'(1,1;1)+I_j''(1,1;1)$$

$$-\frac{2m_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n+1}(m_n+1)\gamma_n}{m_n}I_{jn}(1,1;1)\Bigg]$$

$$-\frac{ia}{2}S_\phi e^{i\phi}\{I_3'(1,1;1)+I_3''(1,1;1)\}. \quad (116)$$

The corresponding results for ring loading buried in an isotropic half space can be found in Appendix I for the three cases of ring loading considered above.

8. DISCUSSION

This analysis has derived the elastic fields caused by concentrated ring loadings in a half space or full space. Solutions for isotropic materials have been previously published but most of the present solutions for transversely isotropic materials appear to be new. The exceptions are the axisymmetric torsion case which has been considered by Erguvan (1987, 1988) and the axisymmetric ring loadings on the surface of a half space analyzed by Hasegawa and Watanabe (1995). The method of solution used here is a direct integration of the derivatives of the potential functions around the circumference of a circle. These integrals were evaluated in terms of complete elliptic integrals. In some instances the derivatives of the potentials were discontinuous functions, an interesting feature not seen previously. However it was shown that these discontinuous functions lead to a continuous elastic field. The continuous part of these derivatives was written in terms of the functions $I(\mu, \nu; \lambda)$, first introduced by Eason *et al.* (1955), to represent an infinite integral containing products of Bessel functions. They evaluated it in terms of complete elliptic integrals for integer μ, ν and λ when the integral was convergent. In the present paper, the notation $I(\mu, \nu; \lambda)$ is used to represent combinations of complete elliptic integrals for the entire body, whether the infinite integral containing products of Bessel functions converges or not.

On a related topic, many of the expressions for the elastic fields derived in this paper contain powers of the radial coordinate ρ in the denominators. However it can be shown that the numerators also vanish leading to a finite result. Perhaps the easiest way to obtain these limits is to use the results in Appendix A of Hanson and Puja (1996b). There the leading terms of the expansions for the functions $I(\mu, \nu; \lambda)$ were given for $\rho \to 0$. These can be used to find the elastic fields for the various loading cases considered in this paper along the z axis where $\rho = 0$. Care must be taken to remember that the expressions in Appendix A of Hanson and Puja (1996b) are only valid for $z > 0$ since they were obtained by expanding the Bessel function from the integral representation in eqn (D1). For $z < 0$, the integral in equation (D1) is meaningless and the expansions in Appendix A of Hanson and Puja (1996b) are no longer valid for the functions $I(\mu, \nu; \lambda)$ which contain the complete elliptic integral of the third kind.

Another interesting feature concerns the symmetry in some of the integrals. For example, consider equation (C8) which is repeated below

$$\int_0^{2\pi}\cos(\phi_0)[z_j\ln[R_j+z_j]-R_j]\,d\phi_0 = 2\pi I_j(1,1;-2)$$

$$+\left\{0, z>0;\ \frac{\pi z_j}{a\rho}(a^2-\rho^2)[1-\text{sgn}(\rho-a)], z<0\right\}. \quad (117)$$

The integrand is symmetric in the variables ρ and a whereas the right hand side does not appear to be. In order to see what happens it is first noted that the terms $l_{1j}(a)$, $l_{2j}(a)$ and thus k_j remain unaltered if one interchanges ρ and a. Therefore $\mathbf{F}(k_j)$ and $\mathbf{E}(k_j)$ are unaffected. However, $\Pi(n_j, k_j)$ is transformed to $\Pi(p_{1j}, k_j)$ where p_{1j} is given as $p_{1j} = [l_{1j}(a)]^2/a^2$. Now a modified version of the transformation formula from Hanson and Puja (1996b) [eqn (26) of that paper] is needed

$$\Pi(p_{1j}, k_j) = \frac{\pi l_{2j}(a)}{2[z_j^2]^{1/2}} + \mathbf{F}(k_j) - \Pi(n_j, k_j). \tag{118}$$

The modifications consist of adding the j subscript and the z in the denominator of the first term on the right side has been replaced with $[z_j^2]^{1/2}$. This change now makes the above equation valid for complex z_j and the square root must be taken as in eqn (21) above. The above change is identical to the modification made to eqn (24) resulting in eqn (28). For $z > 0$, $[z_j^2]^{1/2} = z_j$ and it is easy to show that $I_j(1, 1; -2)$ is symmetric in ρ and a. For $z < 0$, $[z_j^2]^{1/2} = -z_j$ and now the combination of complete elliptic integrals represented by the notation $I_j(1, 1; -2)$ is not symmetric in ρ and a. However the extra discontinuous term in brackets in eqn (117) (with ρ and a now reversed) will make up the difference and the resulting overall expression can be shown symmetric.

The final topic of discussion concerns the discontinuities displayed by some of the functions derived here. For a clear illustration, consider again eqns (A2) and (A8) which provide the result

$$\frac{\partial}{\partial \rho}\psi(\rho, z_j) = \int_0^{2\pi} \frac{\rho - a\cos\phi_0}{R_j(R_j + z_j)} \mathrm{d}\phi_0 = \frac{2\pi}{\rho} - \frac{4z_j}{\rho l_{2j}(a)} \Pi(n_j, k_j) + \left\{0, z > 0; -\frac{2\pi}{\rho}[1 - \mathrm{sgn}(\rho - a)], z < 0\right\}. \tag{119}$$

The integrand is a continuous function for $z > 0$ and also as z passes into the region $z < 0$ for either $\rho > a$ or $\rho < a$. Therefore, this integral should produce a function with the same continuous properties. It is clear that the right hand side is continuous for $z > 0$. Now consider what happens as $z \to 0$ for $\rho > a$. Equation (A11) provides that $\mathrm{Lim}_{z \to 0}\, z_j \Pi(n_j, k_j) = 0$ so the right hand side above gives the value $2\pi/\rho$ as z tends to zero from either positive or negative values and there is continuity. As $z \to 0$ for $\rho < a$, one must consider z positive or z negative. For z positive, eqn (A11) provides $\mathrm{Lim}_{z \to 0}\, z_j \Pi(n_j, k_j) = \pi a/2$ and the right hand side above simplifies to zero (note that $l_2(a)$ becomes a in this limit). For z negative, eqn (A11) provides $\mathrm{Lim}_{z \to 0}\, z_j \Pi(n_j, k_j) = -\pi a/2$ and the first two terms on the right hand side above become $4\pi/\rho$. However the term in brackets is now $-4\pi/\rho$ and the total right hand side is zero. Hence the right hand side above is a continuous function in the region $z > 0$ and into the region $z < 0$ for both $\rho < a$ and $\rho > a$. The only discontinuity on the right hand side above is for $z < 0$ as one passes from $\rho < a$ to $\rho > a$ where a jump of $4\pi/\rho$ occurs.

For a ring normal loading in a full space, the above results leads to the displacement in eqn (63) as

$$u^c = \frac{Qa\,e^{i\phi}}{2A_{44}(m_1 - m_2)} \sum_{j=1}^{2} (-1)^{j+1} \left\{\frac{2\pi}{\rho} - \frac{4z_j}{\rho l_{2j}(a)} \Pi(n_j, k_j)\right\}, \tag{120}$$

since the discontinuous term in brackets in eqn (119) cancelled as it was independent of j. Now the term in brackets in eqn (120) has a discontinuity across the $z = 0$ plane for $\rho < a$. The quantity $z_j \Pi(n_j, k_j)$ jumps from $\pi a/2$ for z slightly positive to $-\pi a/2$ for z slightly negative. However this discontinuity is also independent of j and will cancel out in the summation process. In fact, eqn (120) provides $u^c = 0$ when $z = 0$ for all ρ. This is also correct physically since $z = 0$ is a plane of anti-symmetry in this instance.

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APPENDIX A

Here the derivatives of the potential function $\psi(\rho, z_j)$ are evaluated. It is defined as

$$\psi(\rho, z_j) = \int_0^{2\pi} \ln\left[R_j + z_j\right] \mathrm{d}\phi_0, \quad R_j^2 = \rho^2 + a^2 - 2a\rho\cos(\phi_0) + z_j^2, \quad j = 1, 2, 3. \tag{A1}$$

It was discussed at the end of Section 2 that z_3 is always real whereas z_1 and z_2 are either both real or complex conjugates. It was also shown that Re $\{z_j\} > 0$ if $z > 0$, Re $\{z_j\} < 0$ if $z < 0$ and thus the sign of the real part of z_j is the same as the sign of z itself for $j = 1, 2, 3$.

The authors have presently not evaluated $\psi(\rho, z_j)$ but all necessary derivatives have been found. To start with, the ρ derivative becomes

$$\frac{\partial}{\partial \rho}\psi(\rho, z_j) = \int_0^{2\pi} \frac{\rho - a\cos\phi_0}{R_j(R_j - z_j)} \mathrm{d}\phi_0. \tag{A2}$$

Before an evaluation of this integral is made, an interesting feature should be pointed out. First consider z_j as a real quantity, say z. The denominator is a continuous non-zero function in the region $z > 0$ and hence the integral produces a continuous function in this region. If $z < 0$; then R equals the absolute value of z when $\rho = a$ and $\phi_0 = 0$. Thus $R + z$ vanishes and the denominator is zero. Although the numerator also vanishes, it is seen below that this causes the integral to produce a discontinuity along the semi-infinite cylinder $\rho = a$, $z < 0$. Now consider the complex number z_j. When $\rho = a$ and $\phi_0 = 0$ then R becomes $[z_j^2]^{1/2}$ and two cases can happen [see eqn (21)]. If Re $\{z_j\} > 0$, $R_j + z_j = 2z_j$ and is never zero. If Re $\{z_j\} < 0$, $R_j + z_j = 0$ and the denominator again vanishes. Hence a discontinuity will occur as in the real case. Since the sign of the real part of z_j is the same as the sign of z itself, it may be concluded that for $z > 0$ this integral is a continuous function and when $z < 0$, this integral has a discontinuity at $\rho = a$.

The integral is now evaluated. It can be rewritten as

$$\frac{\partial}{\partial \rho}\psi(\rho, z_j) = \int_0^{2\pi} \frac{[\rho - a\cos\phi_0][R_j - z_j]}{R_j[R_j^2 - z_j^2]} \mathrm{d}\phi_0 = \int_0^{2\pi} \frac{[\rho - a\cos\phi_0]}{[R_j^2 - z_j^2]} \mathrm{d}\phi_0 - z_j \int_0^{2\pi} \frac{[\rho - a\cos\phi_0]}{R_j[R_j^2 - z_j^2]} \mathrm{d}\phi_0. \tag{A3}$$

The first integral is independent of z_j and can be easily evaluated as

$$\int_0^{2\pi} \frac{[\rho - a\cos\phi_0]}{[R_j^2 - z_j^2]} \mathrm{d}\phi_0 = \frac{\pi}{\rho}[1 + \mathrm{sgn}(\rho - a)], \tag{A4}$$

which is discontinuous at $\rho = a$. The second integral can be rewritten as

$$\int_0^{2\pi} \frac{[\rho - a\cos\phi_0]}{R_j[R_j^2 - z_j^2]} \mathrm{d}\phi_0 = \frac{1}{2\rho} I_2 - \frac{(a^2 - \rho^2)}{2\rho} I_3, \quad I_2 = \int_0^{2\pi} \frac{\mathrm{d}\phi_0}{R_j}, \quad I_3 = \int_0^{2\pi} \frac{\mathrm{d}\phi_0}{R_j[R_j^2 - z_j^2]}, \tag{A5}$$

and it is noted that I_2 and I_3 were evaluated by Hanson and Puja (1996a) when z_j was real. For complex z_j, eqn (22) can be applied to these previous results to show

$$I_2 = \frac{4}{(1 + k_j) l_{2j}(a)} \mathbf{F}\left(\frac{2\sqrt{k_j}}{1 + k_j}\right), \quad I_3 = \frac{4}{(1 + k_j) l_{2j}(a)(a + \rho)^2} \Pi\left(p, \frac{2\sqrt{k_j}}{1 + k_j}\right), \quad p = \frac{4a\rho}{(a + \rho)^2}, \tag{A6}$$

which is identical in form to the real case. Combining these results together provides

$$\frac{\partial}{\partial \rho}\psi(\rho, z_j) = \frac{\pi}{\rho}[1 + \mathrm{sgn}(\rho - a)] - \frac{z_j}{2\rho} I_2 + \frac{z_j(a^2 - \rho^2)}{2\rho} I_3. \tag{A7}$$

This form is not very convenient since the integral should be a continuous function for $z > 0$. The first term is discontinuous at $\rho = a$ for all z. As shown in eqn (27) of Section 4, the last term is also discontinuous at $\rho = a$ and the two discontinuities cancel for $z > 0$ whereas they combine for $z < 0$. A better form can be obtained using the transformation formulas (eqns (23) and (28) of Section 4) for the complete elliptic integrals leading to

$$\frac{\partial}{\partial \rho}\psi(\rho, z_j) = \frac{2\pi}{\rho} - \frac{4z_j}{\rho l_{2j}(a)} \Pi(n_j, k_j) + \left\{0, z > 0; -\frac{2\pi}{\rho}[1 - \mathrm{sgn}(\rho - a)], z < 0\right\}.$$

$$= 2\pi I_j(0, 1; 0) + \left\{0, z > 0; -\frac{2\pi}{\rho}[1 - \mathrm{sgn}(\rho - a)], z < 0\right\}. \tag{A8}$$

The functions $I_j(\mu, \nu; \lambda)$ are defined in Appendix D. This form has an advantage since $\Pi(n_j, k_j)$ is a continuous function in both the upper and lower half spaces and hence the discontinuity along the semi-infinite cylinder $\rho = a$, $z < 0$ is isolated to within the term in brackets above. However, it is also noted that the term in brackets is discontinuous across the $z = 0$ plane when $\rho < a$ but $(\partial/\partial\rho)\psi(\rho, z_j)$ should be continuous which it was in eqn (A7).

To understand this it is first noted that

$$\lim_{z \to 0} l_{1j}(a) = \min(a, \rho), \quad \lim_{z \to 0} l_{2j}(a) = \max(a, \rho), \tag{A9}$$

where min is the minimum of the two values and max is the maximum. Thus, k_j and n_j become

$$k_i = \frac{\rho}{a}, \quad \rho < a; \quad k_i = \frac{a}{\rho}, \quad \rho > a, \quad n_i = 1, \quad \rho < a; \quad n_i = \frac{a^2}{\rho^2}, \quad \rho > a. \tag{A10}$$

Thus as $z \to 0$ and $\rho < a$, $n_i \to 1$ and $\Pi(n_i, k_i) \to \infty$. The product $z_i\Pi(n_i, k_i)$ is bounded and eqn (25) along with the identity $z_i^2\rho^2 = [l_{2i}^2(a) - \rho^2][\rho^2 - l_{1i}^2(a)]$ or eqn (28) can be used to show

$$\operatorname*{Lim}_{z\to 0} z_i\Pi(n_i, k_i) = \frac{\pi a z_i}{2[z_i^2]^{1/2}}, \quad \rho < a; \quad \operatorname*{Lim}_{z\to 0} z_i\Pi(n_i, k_i) = 0, \quad \rho > a. \tag{A11}$$

Using this result and eqn (21) it is easily seen that eqn (A8) is continuous across the $z = 0$ plane and in fact vanishes for $z = 0$, $\rho < a$ which is consistent with eqns (A3) and (A4).

Now consider the z_i derivatives of $\psi(\rho, z_i)$. The first one can be written as

$$\frac{\partial}{\partial z_i}\psi(\rho, z_i) = \int_0^{2\pi} \frac{d\phi_0}{R_i} = I_2 = \frac{4}{l_{2i}(a)}\mathbf{F}(k_i) = 2\pi I_i(0, 0; 0). \tag{A12}$$

Applying a second z_i derivative leads to

$$\begin{aligned}\frac{\partial^2}{\partial z_i^2}\psi(\rho, z_i) &= -z_i\int_0^{2\pi} \frac{d\phi_0}{R_i^3} = -\frac{4z_i}{l_{2i}^3(a)(1+k_i)(1-k_i)^2}\mathbf{E}\left(\frac{2\sqrt{k_i}}{1+k_i}\right) \\ &= \frac{4z_i}{l_{2i}^3(a)(1-k_i^2)}\left[\mathbf{F}(k_i) - \frac{2}{1-k_i^2}\mathbf{E}(k_i)\right] = -2\pi I_i(0, 0; 1).\end{aligned} \tag{A13}$$

The remaining derivatives needed can be found as follows. First differentiate eqn (A12) with respect to ρ to obtain

$$\begin{aligned}\frac{\partial^2}{\partial\rho\,\partial z_i}\psi(\rho, z_i) &= \frac{\partial}{\partial\rho}\left\{\frac{4}{l_{2i}(a)}\mathbf{F}(k_i)\right\} \\ &= -\frac{4}{\rho l_{2i}^3(a)(1-k_i^2)}\left[[l_{2i}^2(a) - \rho^2]\mathbf{F}(k_i) - \frac{[z_i^2 + a^2 - \rho^2]}{(1-k_i^2)}\mathbf{E}(k_i)\right] = -2\pi I_i(0\,1; 1),\end{aligned} \tag{A14}$$

where the differentiation was performed using the results in Hanson and Puja (1996b). Finally, since $\psi(\rho, z_i)$ is a harmonic function (apart from a delta function which is presently neglected)

$$\begin{aligned}\frac{\partial^2}{\partial\rho^2}\psi(\rho, z_i) &= \left\{-\frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{\partial^2}{\partial z_i^2}\right\}\psi(\rho, z_i) \\ &= -\frac{2\pi}{\rho^2} - \frac{4z_i}{l_{2i}^3(a)(1-k_i^2)}\left[\mathbf{F}(k_i) - \frac{2}{1-k_i^2}\mathbf{E}(k_i)\right] + \frac{4z_i}{\rho^2 l_{2i}(a)}\Pi(n_i, k_i) \\ &\quad + \left\{0, z > 0; \frac{2\pi}{\rho^2}[1 - \operatorname{sgn}(\rho - a)], z < 0\right\} \\ &= 2\pi\left\{I_i(0, 0; 1) - \frac{1}{\rho}I_i(0, 1; 0)\right\} + \left\{0, z > 0; \frac{2\pi}{\rho^2}[1 - \operatorname{sgn}(\rho - a)], z < 0\right\},\end{aligned} \tag{A15}$$

which is also discontinuous. Dirac delta functions will not be presently included but should be included for integrations of the present solutions.

Using the above derivatives, the following differential operators are easily obtained

$$\Delta\psi(\rho, z_i) = -\frac{\partial^2}{\partial z_i^2}\psi(\rho, z_i) = 2\pi I_i(0, 0; 1), \tag{A16}$$

$$\Lambda^2\psi(\rho, z_i) = e^{i2\phi}\left\{\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho}\right\}\psi(\rho, z_i) = e^{i2\phi}\left[2\pi G_i(0, 0; 1) + \left\{0, z > 0; \frac{4\pi}{\rho^2}[1 - \operatorname{sgn}(\rho - a)], z < 0\right\}\right]. \tag{A17}$$

APPENDIX B

Here the derivatives of the potential function $\Gamma(\rho, z_i)$ are evaluated. It is defined in eqn (43) as

$$\Gamma(\rho, z_i) = \int_0^{2\pi} [z_i \ln [R_i + z_i] - R_i] \, d\phi_0 = z_i \psi(\rho, z_i) - \Phi(\rho, z_i), \quad \Phi(\rho, z_i) = \int_0^{2\pi} R_i \, d\phi_0, \tag{B1}$$

with R_i defined in eqn (A1). As a starting point the ρ derivative of $\Phi(\rho, z_i)$ will be found. It is given as

$$\frac{\partial}{\partial \rho}\Phi(\rho, z_i) = \int_0^{2\pi} \frac{[\rho - a\cos\phi_0]}{R_i} \, d\phi_0 = \frac{1}{2\rho} I_1 - \frac{(a^2 + z_i^2 - \rho^2)}{2\rho} I_2, \quad I_1 = \int_0^{2\pi} R_i \, d\phi_0, \tag{B2}$$

and I_2 was evaluated in Appendix A. The integral I_1 can be evaluated as

$$I_1 = 4(1 + k_i) l_{2i}(a) \mathbf{E}\left(\frac{2\sqrt{k_i}}{1 + k_i}\right). \tag{B3}$$

Using these results and the transformation formulae in eqn (23) leads to

$$\frac{\partial}{\partial \rho}\Phi(\rho, z_i) = \frac{4 l_{2i}(a)}{\rho} \mathbf{E}(k_i) - \frac{4[l_{2i}^2(a) - \rho^2]}{\rho l_{2i}(a)} \mathbf{F}(k_i). \tag{B4}$$

The ρ derivative of $\Gamma(\rho, z_i)$ can now be written as

$$\begin{aligned} \frac{\partial}{\partial \rho}\Gamma(\rho, z_i) &= z_i \frac{\partial}{\partial \rho}\psi(\rho, z_i) - \frac{\partial}{\partial \rho}\Phi(\rho, z_i) \\ &= -2\pi I_i(0, 1; -1) + \left\{0, z > 0; -\frac{2\pi z_i}{\rho}[1 - \operatorname{sgn}(\rho - a)], z < 0\right\}, \end{aligned} \tag{B5}$$

where the discontinuity arises from the radial derivative of $\psi(\rho, z_i)$.

Some additional derivatives can be found as follows. Noting the differential relations

$$\begin{gathered} \frac{\partial}{\partial z_i}\Gamma(\rho, z_i) = \psi(\rho, z_i), \quad \frac{\partial^2}{\partial \rho \, \partial z_i}\Gamma(\rho, z_i) = \frac{\partial}{\partial \rho}\psi(\rho, z_i), \quad \frac{\partial^2}{\partial z_i^2}\Gamma(\rho, z_i) = \frac{\partial}{\partial z_i}\psi(\rho, z_i), \\ \frac{\partial^3}{\partial \rho \, \partial z_i^2}\Gamma(\rho, z_i) = \frac{\partial^2}{\partial \rho \, \partial z_i}\psi(\rho, z_i), \quad \frac{\partial^3}{\partial \rho^2 \, \partial z_i}\Gamma(\rho, z_i) = \frac{\partial^2}{\partial \rho^2}\psi(\rho, z_i), \end{gathered} \tag{B6}$$

it is easy to see that

$$\frac{\partial^2}{\partial \rho \, \partial z_i}\Gamma(\rho, z_i) = 2\pi I_i(0, 1; 0) + \left\{0, z > 0; -\frac{2\pi}{\rho}[1 - \operatorname{sgn}(\rho - a)], z < 0\right\}, \tag{B7}$$

$$\frac{\partial^3}{\partial \rho \, \partial z_i^2}\Gamma(\rho, z_i) = -2\pi I_i(0, 1; 1), \tag{B8}$$

$$\frac{\partial^3}{\partial \rho^2 \, \partial z_i}\Gamma(\rho, z_i) = 2\pi\left\{I_i(0, 0; 1) - \frac{1}{\rho} I_i(0, 1; 0)\right\} + \left\{0, z > 0; \frac{2\pi}{\rho^2}[1 - \operatorname{sgn}(\rho - a)], z < 0\right\}. \tag{B9}$$

Noting that $\Gamma(\rho, z_i)$ is harmonic leads to

$$\begin{aligned} \frac{\partial^2}{\partial \rho^2}\Gamma(\rho, z_i) &= \left\{-\frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{\partial^2}{\partial z_i^2}\right\}\Gamma(\rho, z_i) = -2\pi\left\{I_i(0, 0; 0) - \frac{1}{\rho} I_i(0, 1; -1)\right\} \\ &\quad + \left\{0, z > 0; \frac{2\pi z_i}{\rho^2}[1 - \operatorname{sgn}(\rho - a)], z < 0\right\}. \end{aligned} \tag{B10}$$

The final derivatives that will be evaluated are the operators $\Lambda\Gamma(\rho, z_i)$, $\Delta\Gamma(\rho, z_i)$, $\Lambda\Delta\Gamma(\rho, z_i)$, $\Lambda^2\Gamma(\rho, z_i)$ and $\Lambda^3\Gamma(\rho, z_i)$, where Λ and Δ are defined in eqn (3). The operators $\Lambda\Gamma(\rho, z_i)$, $\Delta\Gamma(\rho, z_i)$ and $\Lambda\Delta\Gamma(\rho, z_i)$ are

$$\Lambda\Gamma(\rho,z_i) = e^{i\phi}\frac{\partial}{\partial\rho}\Gamma(\rho,z_i), \quad \Delta\Gamma(\rho,z_i) = -\frac{\partial^2}{\partial z_i^2}\Gamma(\rho,z_i) = -\frac{\partial}{\partial z_i}\psi(\rho,z_i),$$

$$\Lambda\Delta\Gamma(\rho,z_i) = -e^{i\phi}\frac{\partial^3}{\partial\rho\,\partial z_i^2}\Gamma(\rho,z_i) = -e^{i\phi}\frac{\partial^2}{\partial\rho\,\partial z_i}\psi(\rho,z_i) \tag{B11}$$

and they can be easily obtained from the above results and use of Appendix A. For $\Lambda^2\Gamma(\rho,z_i)$

$$\Lambda^2\Gamma(\rho,z_i) = e^{i2\phi}\left\{\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho}\right\}\Gamma(\rho,z_i) = e^{i2\phi}\left[-2\pi G_i(0,0;0) + \left\{0, z>0; \frac{4\pi z_i}{\rho^2}[1-\operatorname{sgn}(\rho-a)], z<0\right\}\right], \tag{B12}$$

and the operator $\Lambda^3\Gamma(\rho,z_i)$ can be written as

$$\Lambda^3\Gamma(\rho,z_i) = e^{i3\phi}\left\{\frac{\partial^3}{\partial\rho^3} - \frac{3}{\rho}\frac{\partial^2}{\partial\rho^2} + \frac{3}{\rho^2}\frac{\partial}{\partial\rho}\right\}\Gamma(\rho,z_i) = e^{i3\phi}\left\{\frac{4}{\rho}\frac{\partial^2}{\partial z_i^2} + \frac{8}{\rho^2}\frac{\partial}{\partial\rho} - \frac{\partial^3}{\partial\rho\,\partial z_i^2}\right\}\Gamma(\rho,z_i)$$
$$= e^{i3\phi}\left[2\pi H_i(0,1;1) + \left\{0, z>0; -\frac{16\pi z_i}{\rho^3}[1-\operatorname{sgn}(\rho-a)], z<0\right\}\right]. \tag{B13}$$

APPENDIX C

The potential functions $\Omega(\rho,\phi,z_i)$ and $\bar{\Omega}(\rho,\phi,z_i)$ are defined in eqn (51) as

$$\Omega(\rho,\phi,z_i) = \int_0^{2\pi} e^{i\phi_0}[z_i \ln[R_i+z_i] - R_i]\,d\phi_0,$$
$$R_i^2 = \rho^2 + a^2 - 2\rho a\cos(\phi-\phi_0) + z_i^2,$$
$$\bar{\Omega}(\rho,\phi,z_i) = \int_0^{2\pi} e^{-i\phi_0}[z_i \ln[R_i+z_i] - R_i]\,d\phi_0, \tag{C1}$$

and it is noted in this case that the integrals are not independent of ϕ. They cannot be considered as complex conjugate either since z_i can be a complex number. It is easy to show that these integrals can be transformed to

$$\Omega(\rho,\phi,z_i) = e^{i\phi}\int_0^{2\pi} \cos(\phi_0)[z_i \ln[R_i+z_i] - R_i]\,d\phi_0,$$
$$R_i^2 = \rho^2 + a^2 - 2\rho a\cos(\phi_0) + z_i^2,$$
$$\bar{\Omega}(\rho,\phi,z_i) = e^{-i\phi}\int_0^{2\pi} \cos(\phi_0)[z_i \ln[R_i+z_i] - R_i]\,d\phi_0. \tag{C2}$$

To evaluate this integral it is integrated by parts using $\cos\phi_0\,d\phi_0 = d(\sin\phi_0)$ leading to

$$\int_0^{2\pi} \cos(\phi_0)[z_i \ln[R_i+z_i] - R_i]\,d\phi_0 = -\rho a z_i\int_0^{2\pi}\frac{\sin^2\phi_0}{R_i(R_i+z_i)}d\phi_0 + \rho a\int_0^{2\pi}\frac{\sin^2\phi_0}{R_i}d\phi_0. \tag{C3}$$

The first integral can be rewritten as

$$-\rho a z_i\int_0^{2\pi}\frac{\sin^2\phi_0}{R_i(R_i+z_i)}d\phi_0 = -\rho a z_i\int_0^{2\pi}\frac{\sin^2\phi_0[R_i-z_i]}{R_i[R_i^2-z_i^2]}d\phi_0$$
$$= -\rho a z_i\int_0^{2\pi}\frac{\sin^2\phi_0}{[R_i^2-z_i^2]}d\phi_0 + \rho a z_i^2\int_0^{2\pi}\frac{\sin^2\phi_0}{R_i[R_i^2-z_i^2]}d\phi_0. \tag{C4}$$

Using the substitution

$$\sin^2\phi_0 = -\frac{[R_j^2-z_j^2]^2}{4a^2\rho^2}+\frac{(a^2+\rho^2)[R_j^2-z_j^2]}{2a^2\rho^2}-\frac{(a^2-\rho^2)^2}{4a^2\rho^2}, \tag{C5}$$

$$-\rho a z_j\int_0^{2\pi}\frac{\sin^2\phi_0}{R_j(R_j+z_j)}\mathrm{d}\phi_0 = \frac{\pi z_j}{2\rho a}[-(a^2+\rho^2)+(a^2-\rho^2)\,\mathrm{sgn}(a-\rho)]-\frac{z_j^2}{4a\rho}I_1$$

$$+\frac{z_j^2[2a^2+2\rho^2+z_j^2]}{4a\rho}I_2-\frac{z_j^2(a^2-\rho^2)^2}{4a\rho}I_3, \tag{C6}$$

where I_1 was evaluated in Appendix B and I_2, I_3 were evaluated in Appendix A. The Second integral in eqn (C3) was evaluated in Appendix B of Hanson and Puja (1996a) and their result is

$$\rho a\int_0^{2\pi}\frac{\sin^2\phi_0}{R_j}\mathrm{d}\phi_0 = \frac{l_{2j}^2(a)[1+k_j^2]}{6a\rho}I_1-\frac{l_{2j}^4(a)[1-k_j^2]^2}{6a\rho}I_2. \tag{C7}$$

Combining these results and using the transformation formulas in eqns (23) and (28) leads to

$$\int_0^{2\pi}\cos(\phi_0)[z_j\ln[R_j+z_j]-R_j]\,\mathrm{d}\phi_0 = 2\pi I_j(1,1;-2)$$

$$+\left\{0,z>0;\ \frac{\pi z_j}{a\rho}(a^2-\rho^2)[1-\mathrm{sgn}(\rho-a)],z<0\right\}. \tag{C8}$$

Substituting this result into eqn (C2) provides finally

$$\Omega(\rho,\phi,z_j) = \mathrm{e}^{i\phi}f(\rho,z_j),\quad \bar{\Omega}(\rho,\phi,z_j) = \mathrm{e}^{-i\phi}f(\rho,z_j),$$

$$f(\rho,z_j) = \left[2\pi I_j(1,1;-2)+\left\{0,z>0;\frac{\pi z_j}{a\rho}(a^2-\rho^2)[1-\mathrm{sgn}(\rho-a)],z<0\right\}\right]. \tag{C9}$$

Some derivatives of $f(\rho,z_j)$ needed for the elastic field can be found as

$$\left\{\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right\}f(\rho,z_j) = 2\pi I_j(1,0;-1)$$

$$+\left\{0,z>0;-\frac{2\pi z_j}{a}[1-\mathrm{sgn}(\rho-a)],z<0\right\}, \tag{C10}$$

$$\frac{\partial}{\partial z_j}\left\{\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right\}f(\rho,z_j) = -2\pi I_j(1,0;0)+\left\{0,z>0;-\frac{2\pi}{a}[1-\mathrm{sgn}(\rho-a)],z<0\right\}, \tag{C11}$$

$$\frac{\partial^2}{\partial z_j^2}\left\{\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right\}f(\rho,z_j) = 2\pi I_j(1,0;1), \tag{C12}$$

$$\Lambda\left\{\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right\}f(\rho,z_j) = -2\pi\,\mathrm{e}^{i\phi}I_j(1,1;0), \tag{C13}$$

$$\Lambda\frac{\partial}{\partial z_j}\left\{\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right\}f(\rho,z_j) = 2\pi\,\mathrm{e}^{i\phi}I_j(1,1;1), \tag{C14}$$

$$\Lambda^2\left\{\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right\}f(\rho,z_j) = -2\pi\,\mathrm{e}^{i2\phi}G_j(1,0;1). \tag{C15}$$

APPENDIX D

The function $I(\mu,\nu;\lambda)$ was introduced by Eason *et al.* (1955) as the following integral

$$I(\mu,\nu;\lambda) = \int_0^\infty \xi^\lambda J_\mu(a\xi)J_\nu(\rho\xi)\,\mathrm{e}^{-\xi z}\,\mathrm{d}\xi. \tag{D1}$$

Integrals of this type have been evaluated in Erdelyi (1953, 1954a, 1954b) in terms of hypergeometric series and a Legendre function. Eason *et al.* (1955) evaluated this integral for various integer values of μ, ν and λ in terms of the complete elliptic integrals $\mathbf{F}(2\sqrt{k}/1+k)$, $\mathbf{E}(2\sqrt{k}/1+k)$ and Heuman's Lambda function $\Lambda_0(\alpha,\beta)$ which can be given in terms of the complete elliptic integral of the third kind $\Pi[p,(2\sqrt{k}/1+k)]$ where k is defined in eqn (14) and p in eqn (24). The form of the evaluation they provided is inconvenient since it requires different expressions inside or outside the cylinder $\rho = a$ although the function in eqn (D1) is continuous for any $z > 0$. Recently, Hanson and Puja (1996b) re-evaluated this class of integrals in terms of $\mathbf{F}(k)$, $\mathbf{E}(k)$ and $\Pi(n, k)$ where n is also defined in eqn (14). This new form of the evaluations provide a single expression which is continuous everywhere in the half space $z > 0$. The results below are taken from this reference. The functions $I_j(\mu, v; \lambda)$ are obtained from $I(\mu, v; \lambda)$ by substituting $z \to z_j$ in these formulae, $l_2(a) \to l_{2j}(a)$ and $l_1(a) \to l_{1j}(a)$. Thus $I_j(\mu, v; \lambda)$ are given in terms of $\mathbf{F}(k_j)$, $\mathbf{E}(k_j)$ and $\Pi(n_j, k_j)$.

A final comment on this matter is required. The integral above is convergent for $z > 0$ if z is a real quantity. For transversely isotropic materials z is replaced by z_j which may be a complex quantity. Hence the integral is convergent if $\operatorname{Re}\{z_j\} > 0$. As discussed at the end of Section 2, the sign of $\operatorname{Re}\{z_j\}$ is the same as the sign of z itself. Thus for full space problems or ring loads buried in a half space, the above integral will not be convergent in some region. This is of no consequence however since the present analysis evaluates the derivatives of the potential functions by direct integration in terms of a combination of elliptic integrals. These evaluations are valid for any complex z_j and the notation $I_j(\mu, v; \lambda)$ is merely introduced as a shorthand for representing this combination of elliptic integrals. Of course, when $\operatorname{Re}\{z_j\} > 0$, this combination of elliptic integrals will be equal to the integral in eqn (D1) above.

$$I(1,1;-2) = -\frac{za}{2\rho} + \frac{l_2(a)[2\rho^2 + 2a^2 - z^2]}{3\pi a\rho}\mathbf{E}(k) + \frac{l_2^2(a)[4l_1^2(a) + z^2 - 2\rho^2 - 2a^2] + 3z^2\rho^2}{3\pi a\rho l_2(a)}\mathbf{F}(k) + \frac{z^2[a^2 - \rho^2]}{\pi a\rho l_2(a)}\Pi(n,k), \tag{D2}$$

$$I(1,0;-1) = \frac{2}{\pi a l_2(a)}[l_2^2(a)\mathbf{E}(k) - [\rho^2 - l_1^2(a)]\mathbf{F}(k) - z^2\Pi(n,k)], \tag{D3}$$

$$I(0,1;-1) = \frac{2}{\pi\rho l_2(a)}\left[-\frac{z\pi l_2(a)}{2} + l_2^2(a)\mathbf{E}(k) - [l_2^2(a) - \rho^2]\mathbf{F}(k) + z^2\Pi(n,k)\right], \tag{D4}$$

$$I(1,1;-1) = \frac{a}{2\rho} + \frac{zl_2(a)}{\pi a\rho}\mathbf{E}(k) - \frac{z}{\pi a\rho l_2(a)}[\rho^2 + l_2^2(a)]\mathbf{F}(k) - \frac{z(a^2 - \rho^2)}{\pi a\rho l_2(a)}\Pi(n,k), \tag{D5}$$

$$I(0,0;0) = \frac{2}{\pi l_2(a)}\mathbf{F}(k), \tag{D6}$$

$$I(1,0;0) = -\frac{2z}{\pi a l_2(a)}[\mathbf{F}(k) - \Pi(n,k)], \tag{D7}$$

$$I(0,1;0) = \frac{1}{\rho}\left[1 - \frac{2z}{\pi l_2(a)}\Pi(n,k)\right], \tag{D8}$$

$$I(1,1;0) = \frac{2l_2(a)}{\pi a\rho}[\mathbf{F}(k) - \mathbf{E}(k)], \tag{D9}$$

$$I(0,0;1) = \frac{2z}{\pi l_2^3(a)(1-k^2)}\left[-\mathbf{F}(k) + \frac{2}{1-k^2}\mathbf{E}(k)\right], \tag{D10}$$

$$I(1,0;1) = \frac{2[l_2^2(a) - a^2]}{\pi a l_2^3(a)(1-k^2)}\mathbf{F}(k) - \frac{2[z^2 + \rho^2 - a^2]}{\pi a l_2^3(a)(1-k^2)^2}\mathbf{E}(k), \tag{D11}$$

$$I(0,1;1) = \frac{2}{\pi\rho l_2^3(a)(1-k^2)}\left[[l_2^2(a) - \rho^2]\mathbf{F}(k) - \frac{[z^2 + a^2 - \rho^2]}{(1-k^2)}\mathbf{E}(k)\right], \tag{D12}$$

$$I(1,1;1) = \frac{2z}{\pi a\rho l_2(a)(1-k^2)}\left[-\mathbf{F}(k) + \frac{1+k^2}{1-k^2}\mathbf{E}(k)\right], \tag{D13}$$

$$I(0,0;2) = \frac{2\{l_2^2(a)(1-k^2)^2 - 5z^2 - 3z^2k^2\}}{\pi l_2^5(a)(1-k^2)^3}\mathbf{F}(k) - \frac{4\{l_2^2(a)(1-k^2)^2 - 4z^2(1+k^2)\}}{\pi l_2^5(a)(1-k^2)^4}\mathbf{E}(k), \tag{D14}$$

$$I(1,0;2) = \frac{2z\{l_2^2(a) + 7l_1^2(a) - 5a^2 - 3a^2k^2\}}{\pi a l_2^5(a)(1-k^2)^3}\mathbf{F}(k) + \frac{2z\{8a^2(1+k^2) - k^2l_1^2(a) - l_2^2(a) - 14l_1^2(a)\}}{\pi a l_2^5(a)(1-k^2)^4}\mathbf{E}(k), \tag{D15}$$

$$I(0,1;2) = \frac{2z\{l_2^2(a)+7l_1^2(a)-5\rho^2-3\rho^2k^2\}}{\pi\rho l_2^5(a)(1-k^2)^3}\mathbf{F}(k)$$
$$+\frac{2z\{8\rho^2(1+k^2)-k^2l_1^2(a)-l_2^2(a)-14l_1^2(a)\}}{\pi\rho l_2^5(a)(1-k^2)^4}\mathbf{E}(k). \quad \text{(D16)}$$

$$I(1,1;2) = -\frac{2\{-l_2^2(a)(1-k^2)^2+z^2+z^2k^2\}}{\pi a\rho l_2^3(a)(1-k^2)^3}\mathbf{F}(k)$$
$$-\frac{2l_2^2(a)(1+k^2)(1-k^2)^2-2z^2(1+14k^2+k^4)}{\pi a\rho l_2^3(a)(1-k^2)^4}\mathbf{E}(k). \quad \text{(D17)}$$

$$I(0,0;3) = \frac{2z\{(15+9k^2)l_2^2(a)(1-k^2)^2-z^2[31+82k^2+15k^4]\}}{\pi l_2^7(a)(1-k^2)^5}\mathbf{F}(k)$$
$$-\frac{4z\{12(1+k^2)l_2^2(a)(1-k^2)^2-z^2[23+82k^2+23k^4]\}}{\pi l_2^7(a)(1-k^2)^6}\mathbf{E}(k). \quad \text{(D18)}$$

$$I(0,1;3) = -\frac{2A}{\pi\rho l_2^7(a)(1-k^2)^5}\mathbf{F}(k)-\frac{2B}{\pi\rho l_2^7(a)(1-k^2)^6}\mathbf{E}(k),$$
$$A = l_2^2(a)(1-k^2)^2[l_2^2(a)+7l_1^2(a)-5\rho^2-3\rho^2k^2]$$
$$+z^2[-l_2^2(a)(3+74k^2+51k^4)+\rho^2(31+82k^2+15k^4)].$$
$$B = l_2^2(a)(1-k^2)^2[-l_2^2(a)-14l_1^2(a)+8\rho^2-8\rho^2k^2-k^2l_1^2(a)]$$
$$+z^2[3l_2^2(a)+l_1^2(a)(125+125k^2+3k^4)-2\rho^2(23+82k^2+23k^4)]. \quad \text{(D19)}$$

$$I(1,0;3) = -\frac{2C}{\pi al_2^7(a)(1-k^2)^5}\mathbf{F}(k)-\frac{2D}{\pi al_2^7(a)(1-k^2)^6}\mathbf{E}(k),$$
$$C = l_2^2(a)(1-k^2)^2[l_2^2(a)+7l_1^2(a)-5a^2-3a^2k^2]$$
$$+z^2[-l_2^2(a)(3+74k^2+51k^4)+a^2(31+82k^2+15k^4)].$$
$$D = l_2^2(a)(1-k^2)^2[-l_2^2(a)-14l_1^2(a)+8a^2+8a^2k^2-k^2l_1^2(a)]$$
$$+z^2[3l_2^2(a)+l_1^2(a)(125+125k^2+3k^4)-2a^2(23+82k^2+23k^4)]. \quad \text{(D20)}$$

The functions $G(\mu,\nu;\lambda)$ and $H(\mu,\nu;\lambda)$ are given as the following combinations of $\mathbf{I}(\mu,\nu;\lambda)$

$$G(\mu,\nu;\lambda) = I(\mu,\nu;\lambda)-\frac{2}{\rho}I(\mu,\nu+1;\lambda-1), \quad \text{(D21)}$$

$$H(\mu,\nu;\lambda) = I(\mu,\nu;\lambda)+\frac{4}{\rho}I(\mu,\nu-1;\lambda-1)-\frac{8}{\rho^2}I(\mu,\nu;\lambda-2). \quad \text{(D22)}$$

Here the special relations $G(\mu,0;\lambda) = -I(\mu,2;\lambda)$ and $H(\mu,1;\lambda) = -I(\mu,3;\lambda)$ can also be used for some of the functions needed in the elastic fields derived presently. In this regard, some additional results from Hanson and Puja (1996b) are

$$I(0,2;0) = \frac{2}{\pi\rho^2 l_2(a)}[-z\pi l_2(a)+2l_2^2(a)\mathbf{E}(k)+[\rho^2-2l_2^2(a)]\mathbf{F}(k)+2z^2\Pi(n,k)]. \quad \text{(D23)}$$

$$I(0,2;1) = \frac{2}{\rho^2}+\frac{2z}{\pi l_2^3(a)(1-k^2)}\left[\mathbf{F}(k)-\frac{2}{1-k^2}\mathbf{E}(k)\right]-\frac{4z}{\pi\rho^2 l_2(a)}\Pi(n,k). \quad \text{(D24)}$$

$$I(0,2;2) = \frac{2\{l_2^2(a)[2l_2^2(a)-3\rho^2](1-k^2)^2-z^2\rho^2(5+3k^2)\}}{\pi\rho^2 l_2^5(a)(1-k^2)^3}\mathbf{F}(k)$$
$$+\frac{4\{l_2^2(a)[2\rho^2-a^2-z^2](1-k^2)^2-4\rho^2z^2(1+k^2)\}}{\pi\rho^2 l_2^5(a)(1-k^2)^4}\mathbf{E}(k). \quad \text{(D25)}$$

$$I(1,2;1) = \frac{2[2l_2^2(a)-l_1^2(a)-\rho^2]}{\pi a\rho^2 l_2(a)(1-k^2)}\mathbf{F}(k)+\frac{2[\rho^2(z^2+\rho^2-a^2)-2(l_2^2(a)-l_1^2(a))^2]}{\pi a\rho^2 l_2^3(a)(1-k^2)^2}\mathbf{E}(k), \tag{D26}$$

$$I(1,2;2) = -\frac{2z\{2l_2^2(a)(1-k^2)^2+7\rho^2k^2+\rho^2-5l_1^2(a)-3l_1^2(a)k^2\}}{\pi a\rho^2 l_2^3(a)(1-k^2)^3}\mathbf{F}(k)$$
$$+\frac{2z\{2(1+k^2)[l_2^2(a)-6l_1^2(a)+l_1^2(a)k^2]+\rho^2[1+k^4+14k^2]\}}{\pi a\rho^2 l_2^3(a)(1-k^2)^4}\mathbf{E}(k). \tag{D27}$$

For the functions $H(\mu,1;\lambda)$, the eqn (D22) must be used since the functions $I(\mu,3;\lambda)$ have not been evaluated by the authors. Similarly, eqn (D21) must be used for $G(\mu,0;3)$ since $I(\mu,2;3)$ have likewise not been evaluated.

APPENDIX E

The potential functions for a buried load in a transversely isotropic half space have been given by Shield (1951), Fabrikant (1970) and Pan and Chow (1979). The results from Fabrikant (1970) are re-derived here for convenience.

Point normal force

To start with, two point forces in a full space are used. One at the point (ρ_0,ϕ_0,h) in the positive z direction with a mangitude P and one at $(\rho_0,\phi_0,-h)$ in the negative z direction, also with magnitude P. The potentials for these two loads can be written directly from eqn (60) as

$$F_j(\rho,\phi,z;\rho_0,\phi_0) = \frac{(-1)^{j+1}P}{4\pi A_{44}(m_1-m_2)}[\ln[R_j'-z_j']-\ln[R_j''+z_j'']],\quad j=1,2,\quad F_3(\rho,\phi,z;\rho_0,\phi_0)=0,$$
$$z'=z-h,\quad z_j'=\frac{z'}{\gamma_j}=z_j-h_j,\quad z''=z+h,\quad z_j''=\frac{z''}{\gamma_j}=z_j+h_j,\quad h_j=\frac{h}{\gamma_j},$$
$$R_j'^2=\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z_j'^2,\quad R_j''^2=\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+z_j''^2. \tag{E1}$$

The plane $z=0$ is then one of symmetry on which the shear stress τ_z is automatically zero. There exists a normal stress which is denoted as σ_{zz}' which can be found from eqn (7) as

$$\sigma_{zz}'=\frac{P}{2\pi(m_1-m_2)}\sum_1^2(-1)^{j+1}(1+m_j)\frac{h_j}{(\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+h_j^2)^{3/2}}. \tag{E2}$$

This normal traction must be removed. To do so, the following potentials are introduced

$$F_j(\rho,\phi,z)=\frac{(-1)^{j+1}\gamma_j}{(m_j+1)}F(\rho,\phi,z_j),\quad j=1,2,\quad F_3(\rho,\phi,z)=0. \tag{E3}$$

This form automatically satisfies zero shear stress on the $z=0$ plane. The function $F(\rho,\phi,z)$ must be harmonic with no singularities in the region $z>0$. The function $F(\rho,\phi,z)$ is such that the normal stress on the $z=0$ plane resulting from these new potentials cancels that in eqn (E2) leading to the result

$$\operatorname*{Lim}_{z\to 0}\frac{\partial^2}{\partial z^2}F(\rho,\phi,z)=-\frac{\sigma_{zz}'}{A_{44}(\gamma_1-\gamma_2)}. \tag{E4}$$

Choosing the function $F(\rho,\phi,z)$ based on the form in eqn (E2) leads to

$$F(\rho,\phi,z)=\frac{P}{2\pi(m_1-m_2)A_{44}(\gamma_1-\gamma_2)}\sum_{n=1}^2(-1)^{n+1}(1+m_n)\ln[[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+(z+h_n)^2]^{1/2}+z+h_n],$$

$$\frac{\partial^2}{\partial z^2}F(\rho,\phi,z)=\frac{P}{2\pi(m_1-m_2)A_{44}(\gamma_1-\gamma_2)}\sum_{n=1}^2(-1)^{n+1}(1+m_n)\frac{-(z+h_n)}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+(z+h_n)^2]^{3/2}}. \tag{E5}$$

It is easy to see that $F(\rho,\phi,z)$ is harmonic and non-singular in the region $z>0$. Furthermore its second z derivative satisfies eqn (E4). Substituting eqn (E5) into eqn (E3) provides

$$F_j(\rho,\phi,z)=\frac{(-1)^{j+1}\gamma_j}{(m_j+1)}\frac{P}{2\pi(m_1-m_2)A_{44}(\gamma_1-\gamma_2)}\sum_{n=1}^2(-1)^{n+1}(1+m_n)\ln[R_{jn}+(z_j+h_n)],\quad j=1,2,$$
$$R_{jn}^2=\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+(z_j+h_n)^2. \tag{E6}$$

Adding the potentials in eqns (E1) and (E6) leads to the result given in eqn (93).

Point tangential force

In this case a point force with components T_x and T_y in the positive x and y directions at the point (ρ_0, ϕ_0, h) and a symmetric force at $(\rho_0, \phi_0, -h)$ also with components T_x and T_y in the positive x and y directions are used. The potentials for these two loads can be written directly from eqn (61) as

$$F_j(\rho, \phi, z; \rho_0, \phi_0) = \frac{(-1)^{j+1}\gamma_j}{8\pi A_{44}(m_1-m_2)m_j}(T\bar{\Lambda}+\bar{T}\Lambda)[\chi(z_j')+\chi(z_j'')], \quad j = 1, 2$$

$$F_3(\rho, \phi, z; \rho_0, \phi_0) = \frac{i\gamma_3}{8\pi A_{44}}(T\bar{\Lambda}-\bar{T}\Lambda)[\chi(z_3')+\chi(z_3'')]. \tag{E7}$$

The normal stress on the $z = 0$ plane of symmetry is again denoted as σ_{zz}' and can be found as

$$\sigma_{zz}' = \frac{1}{4\pi(m_1-m_2)}\sum_{1}^{2}\frac{(-1)^{j+1}\gamma_j(1+m_j)}{m_j}(T\bar{\Lambda}+\bar{T}\Lambda)\frac{1}{(\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+h_j^2)^{1/2}}. \tag{E8}$$

To remove this normal stress a second set of potentials are taken in the form of eqn (E3) where the harmonic function $F(\rho, \phi, z)$ must satisfy eqn (E4) with σ_{zz}' as now given in eqn (E8). Such a function can be found in the form

$$F(\rho, \phi, z) = \frac{-1}{4\pi(m_1-m_2)A_{44}(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n+1}\gamma_n(1+m_n)}{m_n}(T\bar{\Lambda}+\bar{T}\Lambda)\chi(z+h_n),$$

$$\frac{\partial^2}{\partial z^2}F(\rho, \phi, z) = \frac{-1}{4\pi(m_1-m_2)A_{44}(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n+1}\gamma_n(1+m_n)}{m_n}(T\bar{\Lambda}+\bar{T}\Lambda)$$

$$\times\frac{1}{[\rho^2+\rho_0^2-2\rho\rho_0\cos(\phi-\phi_0)+(z+h_n)^2]^{1/2}}. \tag{E9}$$

In this case the second set of potentials become

$$F_j(\rho, \phi, z) = \frac{(-1)^{j+1}\gamma_j}{(m_j+1)}\frac{-1}{4\pi(m_1-m_2)A_{44}(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n+1}\gamma_n(1+m_n)}{m_n}(T\bar{\Lambda}+\bar{T}\Lambda)\chi(z_j+h_n),$$

$$\chi(z_j+h_n) = (z_j+h_n)\ln[R_{jn}+(z_j+h_n)]-R_{jn}. \tag{E10}$$

Adding the potentials in eqns (E7) and (E10) leads to the results in eqn (94).

APPENDIX F

The quadratic eqn (5) can be written in terms of compliance S_{ij} as

$$am_j^2+bm_j+c = 0, \quad a = S_{11}S_{33}-S_{13}^2, \quad b = 2S_{13}(S_{12}-S_{11})-S_{11}S_{44}, \quad c = S_{11}^2-S_{12}^2. \tag{F1}$$

For real solutions to exist, $b^2-4ac > 0$. This implies the absolute value of b satisfies $|b| > (b^2-4ac)^{1/2}$, since it will be shown below that $a > 0$, $c > 0$. Thus, if b is negative two positive real roots will exist. There will be two negative real roots if $b^2-4ac > 0$ and $b > 0$. Thus the case of one real positive and one real negative root will not occur.

Some restrictions on the elastic constants will now be considered. These have been given by Jones (1975) for an orthotropic material. For a transversely isotropic material it is noted that $S_{55} = S_{44}$, $S_{22} = S_{11}$, $S_{23} = S_{13}$ and $S_{66} = 2(S_{11}-S_{12})$. The restrictions are

$$S_{11}, S_{33}, S_{44}, S_{66} > 0, \quad S_{11}^2-S_{12}^2 > 0, \quad S_{11}S_{33}-S_{13}^2 > 0. \tag{F2}$$

where the inequalities above ensure $a > 0$ and $c > 0$. The above results also imply that $S_{11}-S_{12} > 0$ and $S_{11}+S_{12} > 0$. A final inequality which will be needed is

$$\begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{11} & S_{13} \\ S_{13} & S_{13} & S_{33} \end{vmatrix} = (S_{11}-S_{12})(S_{11}S_{33}+S_{12}S_{33}-2S_{13}^2) > 0, \tag{F3}$$

which leads to

$$S_{11}S_{33}+S_{12}S_{33}-2S_{13}^2 > 0. \tag{F4}$$

The case for concern is when $b^2-4ac > 0$ and $b > 0$ leading to two negative real roots. Assuming this to be true leads to $b > 2\sqrt{(ac)}$ which can be written using eqn (F1) as

$$2S_{13}(S_{12}-S_{11})-S_{11}S_{44} > 2(S_{11}S_{33}-S_{13}^2)^{1/2}(S_{11}^2-S_{12}^2)^{1/2}, \tag{F5}$$

which implies $S_{13} < 0$ and the following steps

$$2S_{13}(S_{12}-S_{11}) > 2(S_{11}S_{33}-S_{13}^2)^{1/2}(S_{11}^2-S_{12}^2)^{1/2}$$

$$S_{13}^2(S_{12}-S_{11})^2 > (S_{11}S_{33}-S_{13}^2)(S_{11}^2-S_{12}^2)$$

$$S_{13}^2(S_{11}-S_{12}) > (S_{11}S_{33}-S_{13}^2)(S_{11}+S_{12})$$

$$S_{11}[2S_{13}^2-S_{33}(S_{11}+S_{12})] > 0. \tag{F6}$$

Since $S_{11} > 0$, the term in brackets above must be positive also. However this term is always negative by the restriction in eqn (F4). Hence, the above assumption leads to a contradiction and this case can never occur. Therefore if $b^2-4ac > 0$ producing real roots, b must be negative. Using the result $S_{11}-S_{12} > 0$, b will always be negative if $S_{13} > 0$. More generally, if $b^2-4ac > 0$ then b will be negative if S_{13} satisfies the inequality

$$S_{13} > -\frac{S_{11}S_{44}}{2(S_{11}-S_{12})}. \tag{F7}$$

APPENDIX G

The solutions for ring loading on an isotropic half space are given below. Here μ is the shear modulus and ν is the Poisson ratio. These expressions can be obtained from Section 5 using the isotropic limits provided in Fabrikant (1991) and Hanson and Puja (1995a).

Ring normal loading

$$u^c = -\frac{Qa\,e^{i\phi}}{2\mu}[(1-2\nu)I(0,1;0)-zI(0,1;1)], \tag{G1}$$

$$w = \frac{Qa}{2\mu}[2(1-\nu)I(0,0;0)+zI(0,0;1)], \tag{G2}$$

$$\sigma_1 = -aQ[(1+2\nu)I(0,0;1)-zI(0,0;2)], \tag{G3}$$

$$\sigma_{zz} = -aQ[I(0,0;1)+zI(0,0;2)], \tag{G4}$$

$$\sigma_2 = -aQ\,e^{i2\phi}[(1-2\nu)G(0,0;1)-zG(0,0;2)], \tag{G5}$$

$$\tau_z = -aQ\,e^{i\phi}zI(0,1;2). \tag{G6}$$

Ring shear loading in the x and y directions

$$u^c = \frac{a}{4\mu}[S\{2(2-\nu)I(0,0;0)-zI(0,0;1)\}-\bar{S}e^{i2\phi}\{2\nu G(0,0;0)+zG(0,0;1)\}], \tag{G7}$$

$$w = \frac{a}{2\mu}[S_x\cos\phi+S_y\sin\phi][(1-2\nu)I(0,1;0)+zI(0,1;1)], \tag{G8}$$

$$\sigma_1 = -a[S_x\cos\phi+S_y\sin\phi][2(1+\nu)I(0,1;1)-zI(0,1;2)], \tag{G9}$$

$$\sigma_{zz} = -a[S_x\cos\phi+S_y\sin\phi]zI(0,1;2), \tag{G10}$$

$$\sigma_2 = -\frac{a}{2}[S e^{i\phi}\{2(2-\nu)I(0,1;1)-zI(0,1;2)\}-\bar{S}e^{i3\phi}\{2\nu H(0,1;1)+zH(0,1;2)\}], \tag{G11}$$

$$\tau_z = -\frac{a}{2}[S\{2I(0,0;1)-zI(0,0;2)\}-z\bar{S}e^{i2\phi}G(0,0;2)]. \tag{G12}$$

Ring shear loading in the ρ and ϕ directions

$$u^c = \frac{a}{2\mu}S_\rho e^{i\phi}[2(1-\nu)I(1,1;0)-zI(1,1;1)]+\frac{ai}{\mu}S_\phi e^{i\phi}I(1,1;0), \tag{G13}$$

$$w = -\frac{a}{2\mu}S_\rho[(1-2\nu)I(1,0;0)+zI(1,0;1)], \tag{G14}$$

$$\sigma_1 = aS_\rho[2(1+\nu)I(1,0;1)-zI(1,0;2)], \tag{G15}$$

$$\sigma_{zz} = aS_\rho zI(1,0;2). \tag{G16}$$

$$\sigma_2 = aS_\rho\, \mathrm{e}^{i2\phi}[2(1-\nu)G(1,0;1) - zG(1,0;2)] + 2iaS_\phi\, \mathrm{e}^{i2\phi}G(1,0;1), \tag{G17}$$

$$\tau_z = -aS_\rho\, \mathrm{e}^{i\phi}[I(1,1;1) - zI(1,1;2)] - iaS_\phi\, \mathrm{e}^{i\phi}I(1,1;1). \tag{G18}$$

APPENDIX H

The solutions for ring loading in an isotropic full space are given below. The solution for isotropy is obtained by a limiting form of the transversely isotropic results in Section 6. For isotropy $\gamma_1, \gamma_2, \gamma_3, m_1, m_2 \to 1$ (Fabrikant, 1989) and each term in the summation becomes indeterminate. A limiting procedure is thus required. The isotropic limits of the two term summations are given below. Here μ is the shear modulus and ν is the Poisson ratio.

$$\sum_1^2 (-1)^{j+1} f(z_j) = -\frac{zf'(z)}{4(1-\nu)}(m_1 - m_2), \tag{H1}$$

$$\sum_1^2 \frac{(-1)^{j+1} m_j}{\gamma_j} f(z_j) = \frac{(3-4\nu)f(z) - zf'(z)}{4(1-\nu)}(m_1 - m_2), \tag{H2}$$

$$\sum_1^2 (-1)^{j+1}(1+m_j) f(z_j) = \frac{2(1-\nu)f(z) - zf'(z)}{2(1-\nu)}(m_1 - m_2), \tag{H3}$$

$$\sum_1^2 \frac{(-1)^{j+1}(1+m_j)}{\gamma_j} f(z_j) = \frac{(1-2\nu)f(z) - zf'(z)}{2(1-\nu)}(m_1 - m_2), \tag{H4}$$

$$\sum_1^2 \frac{(-1)^{j+1}(1+m_j)}{\gamma_j^2} f(z_j) = -\frac{2\nu f(z) + zf'(z)}{2(1-\nu)}(m_1 - m_2), \tag{H5}$$

$$\sum_1^2 \frac{(-1)^{j+1}[\gamma_j^2 - (1+m_j)\gamma_3^2]}{\gamma_j^2} f(z_j) = \frac{4\nu f(z) + zf'(z)}{4(1-\nu)}(m_1 - m_2), \tag{H6}$$

$$\sum_1^2 \frac{(-1)^{j+1}\gamma_j}{m_j} f(z_j) = -\frac{(3-4\nu)f(z) + zf'(z)}{4(1-\nu)}(m_1 - m_2), \tag{H7}$$

$$\sum_1^2 \frac{(-1)^{j+1}[\gamma_j^2 - (1+m_j)\gamma_3^2]}{m_j\gamma_j} f(z_j) = \frac{3f(z) + zf'(z)}{4(1-\nu)}(m_1 - m_2), \tag{H8}$$

$$\sum_1^2 \frac{(-1)^{j+1}(1+m_j)\gamma_j}{m_j} f(z_j) = -\frac{(1-2\nu)f(z) + zf'(z)}{2(1-\nu)}(m_1 - m_2), \tag{H9}$$

$$\sum_1^2 \frac{(-1)^{j+1}(1+m_j)}{m_j} f(z_j) = -\frac{2(1-\nu)f(z) + zf'(z)}{2(1-\nu)}(m_1 - m_2). \tag{H10}$$

Ring normal loading

$$u^c = \frac{Qa\, \mathrm{e}^{i\phi}}{8\mu(1-\nu)} zI(0,1;1), \tag{H11}$$

$$w = \frac{Qa}{8\mu(1-\nu)}[(3-4\nu)I(0,0;0) + zI(0,0;1)], \tag{H12}$$

$$\sigma_1 = -\frac{Qa}{4(1-\nu)}[4\nu I(0,0;1) - zI(0,0;2)], \tag{H13}$$

$$\sigma_{zz} = -\frac{Qa}{4(1-\nu)}[2(1-\nu)I(0,0;1) + zI(0,0;2)], \tag{H14}$$

$$\sigma_2 = \frac{Qa\, \mathrm{e}^{i2\phi}}{4(1-\nu)} zG(0,0;2), \tag{H15}$$

$$\tau_z = -\frac{Qa\,e^{i\phi}}{4(1-\nu)}[(1-2\nu)I(0,1;1)+zI(0,1;2)]. \tag{H16}$$

Ring shear loading in the x *and* y *directions*

$$u^c = \frac{a}{16\mu(1-\nu)}[S\{(7-8\nu)I(0,0;0)-zI(0,0;1)\}-\bar{S}e^{i2\phi}\{G(0,0;0)+zG(0,0;1)\}], \tag{H17}$$

$$w = \frac{a}{8\mu(1-\nu)}[S_x\cos\phi+S_y\sin\phi]zI(0,1;1), \tag{H18}$$

$$\sigma_1 = -\frac{a}{4(1-\nu)}[S_x\cos\phi+S_y\sin\phi][3I(0,1;1)-zI(0,1;2)]. \tag{H19}$$

$$\sigma_{zz} = \frac{a}{4(1-\nu)}[S_x\cos\phi+S_y\sin\phi][(1-2\nu)I(0,1;1)-zI(0,1;2)], \tag{H20}$$

$$\sigma_2 = -\frac{a}{8(1-\nu)}[Se^{i\phi}\{(7-8\nu)I(0,1;1)-zI(0,1;2)\}-\bar{S}e^{i3\phi}\{H(0,1;1)+zH(0,1;2)\}], \tag{H21}$$

$$\tau_z = -\frac{a}{8(1-\nu)}[S\{4(1-\nu)I(0,0;1)-zI(0,0;2)\}-z\bar{S}e^{i2\phi}G(0,0;2)]. \tag{H22}$$

Ring shear loading in the ρ *and* ϕ *directions*

$$u^c = \frac{a}{8\mu(1-\nu)}S_\rho e^{i\phi}[(3-4\nu)I(1,1;0)-zI(1,1;1)]+\frac{ai}{2\mu}S_\phi e^{i\phi}I(1,1;0), \tag{H23}$$

$$w = -\frac{a}{8\mu(1-\nu)}S_\rho zI(1,0;1), \tag{H24}$$

$$\sigma_1 = \frac{a}{4(1-\nu)}S_\rho[3I(1,0;1)-zI(1,0;2)], \tag{H25}$$

$$\sigma_{zz} = -\frac{a}{4(1-\nu)}S_\rho[(1-2\nu)I(1,0;1)-zI(1,0;2)], \tag{H26}$$

$$\sigma_2 = \frac{a}{4(1-\nu)}S_\rho e^{i2\phi}[(3-4\nu)G(1,0;1)-zG(1,0;2)]+iaS_\phi e^{i2\phi}G(1,0;1), \tag{H27}$$

$$\tau_z = -\frac{a}{4(1-\nu)}S_\rho e^{i\phi}[2(1-\nu)I(1,1;1)-zI(1,1;2)]-\frac{ia}{2}S_\phi e^{i\phi}I(1,1;1). \tag{H28}$$

The displacements produced by the ring normal loading Q and the ring shear loads S_ρ and S_ϕ are in agreement with those derived by Kermanidis (1975). It is noted that his parameter e is ρ at present and his radius of loading ρ is denoted here as a. The eqn (23) must also be used since his results are given in terms of $\mathbf{F}(\kappa)$ and $\mathbf{E}(\kappa)$ where $\kappa = 2\sqrt{k}/1+k$. It is noted that the expression for the radial displacement u in his eqn (2.4) contains a misprint in the denominator for the first term on the right hand side where $\sqrt{((e+\rho)^2-z^2)}$ should read $\sqrt{((e+\rho)^2+z^2)}$. In this same equation the number 3·5 should be interpreted as 3.5 and not 15. Finally, before his eqn (2.8) the intensity of loading should read $2\pi R_\phi \rho$ rather than $2\pi R_\phi \rho^2$.

The present isotropic results for displacements also agree with Hasegawa (1992a) for the same cases of normal loading Q and the ring shear loads S_ρ and S_ϕ. Since Hasegawa considers unit body forces, one must replace the intensities Q, S_ρ and S_ϕ with $(2\pi a)^{-1}$. For comparative purposes, one can use Section 12 of Hanson and Puja (1996b) which discusses the relations between the Legendre functions $Q_{-1/2}(x)$, $Q_{1/2}(x)$ and the complete elliptic integrals $\mathbf{F}(k)$ and $\mathbf{E}(k)$. Some pertinent results needed for comparison are (where x is defined by Hasegawa and k is defined in eqn (14))

$$Q_{-1/2}(x) = 2\sqrt{k}\mathbf{F}(k),\quad Q_{1/2}(x) = \frac{2}{\sqrt{k}}[\mathbf{F}(k)-\mathbf{E}(k)],\quad x = \frac{1+k^2}{2k},\quad \frac{1}{x^2-1} = \frac{4k^2}{(1-k^2)^2},$$

$$G_1(x) = \frac{1}{k\sqrt{k}}[(1-k^2)\mathbf{F}(k)-(1+k^2)\mathbf{E}(k)],\quad G_2(x) = \frac{1}{\sqrt{k}}[2\mathbf{E}(k)-(1-k^2)\mathbf{F}(k)]. \tag{H29}$$

APPENDIX I

The solutions for ring loading in an isotropic half space are given below. Here μ is the shear modulus and ν is the Poisson ratio. These expressions can be obtained from Section 7 using the isotropic limits given in Appendix H and the limits for the double sums below.

$$\frac{1}{(m_1-m_2)}\sum_{j=1}^{2}\frac{(-1)^{j+1}\gamma_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}(-1)^{n+1}(m_n+1)f(z_j+h_n)$$
$$=\frac{-1}{4(1-\nu)}[2(1-\nu)(1-2\nu)f(z+h)+\{2z(1-\nu)-h(1-2\nu)\}f'(z+h)-zhf''(z+h)]. \quad \text{(I1)}$$

$$\frac{1}{(m_1-m_2)}\sum_{j=1}^{2}\frac{(-1)^{j-1}m_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}(-1)^{n-1}(m_n+1)f(z_j+h_n)$$
$$=\frac{1}{4(1-\nu)}[4(1-\nu)^2f(z+h)-2(1-\nu)(z+h)f'(z+h)+zhf''(z+h)]. \quad \text{(I2)}$$

$$\frac{1}{(m_1-m_2)}\sum_{j=1}^{2}\frac{(-1)^{j+1}[\gamma_j^2-(1+m_j)\gamma_3^2]}{\gamma_j(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}(-1)^{n+1}(m_n+1)f(z_j+h_n)$$
$$=\frac{1}{4(1-\nu)}[2(1-\nu)(1+2\nu)f(z+h)+\{2z(1-\nu)-h(1+2\nu)\}f'(z+h)-zhf''(z+h)]. \quad \text{(I3)}$$

$$\frac{1}{(m_1-m_2)}\sum_{j=1}^{2}\frac{(-1)^{j-1}\gamma_j}{(\gamma_1-\gamma_2)}\sum_{n=1}^{2}(-1)^{n-1}(m_n+1)f(z_j+h_n)$$
$$=\frac{1}{2(1-\nu)}[2(1-\nu)f(z+h)-\{2z(1-\nu)+h\}f'(z+h)+zhf''(z+h)]. \quad \text{(I4)}$$

$$\frac{1}{(m_1-m_2)}\sum_{j=1}^{2}\frac{(-1)^{j+1}}{(\gamma_1-\gamma_2)}\sum_{n=1}^{2}(-1)^{n+1}(m_n+1)f(z_j+h_n)=\frac{-z}{2(1-\nu)}[2(1-\nu)f'(z+h)-hf''(z+h)]. \quad \text{(I5)}$$

$$\frac{1}{(m_1-m_2)}\sum_{j=1}^{2}\frac{(-1)^{j+1}\gamma_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n+1}(m_n+1)\gamma_n}{m_n}f(z_j+h_n)$$
$$=\frac{1}{4(1-\nu)}[(1-2\nu)^2f(z+h)+(1-2\nu)(z+h)f'(z+h)+zhf''(z+h)]. \quad \text{(I6)}$$

$$\frac{1}{(m_1-m_2)}\sum_{j=1}^{2}\frac{(-1)^{j-1}m_j}{(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n+1}(m_n+1)\gamma_n}{m_n}f(z_j+h_n)$$
$$=\frac{-1}{4(1-\nu)}[2(1-\nu)(1-2\nu)f(z+h)+\{2(1-\nu)h-z(1-2\nu)\}f'(z+h)-zhf''(z+h)]. \quad \text{(I7)}$$

$$\frac{1}{(m_1-m_2)}\sum_{j=1}^{2}\frac{(-1)^{j+1}[\gamma_j^2-(1+m_j)\gamma_3^2]}{\gamma_j(m_j+1)(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n+1}(m_n+1)\gamma_n}{m_n}f(z_j+h_n)$$
$$=\frac{-1}{4(1-\nu)}[(1+2\nu)(1-2\nu)f(z+h)+\{z(1-2\nu)+h(1+2\nu)\}f'(z+h)+zhf''(z+h)]. \quad \text{(I8)}$$

$$\frac{1}{(m_1-m_2)}\sum_{j=1}^{2}\frac{(-1)^{j+1}\gamma_j}{(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n-1}(m_n+1)\gamma_n}{m_n}f(z_j+h_n)$$
$$=\frac{1}{2(1-\nu)}[-(1-2\nu)f(z+h)+\{z(1-2\nu)-h\}f'(z+h)+zhf''(z+h)]. \quad \text{(I9)}$$

$$\frac{1}{(m_1-m_2)}\sum_{j=1}^{2}\frac{(-1)^{j+1}}{(\gamma_1-\gamma_2)}\sum_{n=1}^{2}\frac{(-1)^{n-1}(m_n+1)\gamma_n}{m_n}f(z_j+h_n)=\frac{z}{2(1-\nu)}[(1-2\nu)f'(z+h)+hf''(z+h)]. \quad \text{(I10)}$$

The following notation is also used below.

$$I'(\mu,\nu;\lambda)=I(\mu,\nu;\lambda),\quad z\to z';\quad I''(\mu,\nu;\lambda)=I(\mu,\nu;\lambda),\quad z\to z'',\quad z'=z-h;\quad z''=z+h. \quad \text{(I11)}$$

Ring normal loading

$$u^c=\frac{-Qa\,e^{i\phi}}{8\mu(1-\nu)}[-z'I'(0,1;1)-(3-4\nu)(z-h)I''(0,1;1)+4(1-\nu)(1-2\nu)I''(0,1;0)-2hzI''(0,1;2)], \quad \text{(I12)}$$

$$w=\frac{Qa}{8\mu(1-\nu)}[(3-4\nu)I'(0,0;0)+z'I'(0,0;1)+(5-12\nu+8\nu^2)I''(0,0;0)$$
$$+z''(3-4\nu)I''(0,0;1)+2hzI''(0,0;2)]. \quad \text{(I13)}$$

$$\sigma_1 = -\frac{Qa}{4(1-\nu)}[4\nu I'(0,0;1)+4(1-2\nu^2)I''(0,0;1)-z'I'(0,0;2)$$
$$+\{h(3+4\nu)-z(3-4\nu)\}I''(0,0;2)-2hzI''(0,0;3)], \tag{I14}$$

$$\sigma_{zz} = -\frac{Qa}{4(1-\nu)}[2(1-\nu)I'(0,0;1)+2(1-\nu)I''(0,0;1)+z'I'(0,0;2)$$
$$+\{h+z(3-4\nu)\}I''(0,0;2)+2hzI''(0,0;3)], \tag{I15}$$

$$\sigma_2 = -\frac{Qa\,e^{i2\phi}}{4(1-\nu)}[-z'G'(0,0;2)+4(1-\nu)(1-2\nu)G''(0,0;1)$$
$$-(3-4\nu)(z-h)G''(0,0;2)-2hzG''(0,0;3)], \tag{I16}$$

$$\tau_z = -\frac{Qa\,e^{i2\phi}}{4(1-\nu)}[(1-2\nu)I'(0,1;1)-(1-2\nu)I''(0,1;1)+z'I'(0,1;2)$$
$$+\{z(3-4\nu)-h\}I''(0,1;2)+2hzI''(0,1;3)]. \tag{I17}$$

Ring shear loading in the x *and* y *directions*

$$u^c = \frac{a}{16\mu(1-\nu)}[S\{(7-8\nu)I'(0,0;0)+(9-16\nu+8\nu^2)I''(0,0;0)$$
$$-z'I'(0,0;1)-z''(3-4\nu)I''(0,0;1)+2hzI''(0,0;2)\}$$
$$-\bar{S}e^{i2\phi}\{G'(0,0;0)-(1-8\nu+8\nu^2)G''(0,0;0)+z'G'(0,0;1)$$
$$+z''(3-4\nu)G''(0,0;1)-2hzG''(0,0;2)\}], \tag{I18}$$

$$w = \frac{a}{8\mu(1-\nu)}[S_x\cos\phi+S_y\sin\phi][z'I'(0,1;1)+4(1-\nu)(1-2\nu)I''(0,1;0)$$
$$+(3-4\nu)(z-h)I''(0,1;1)-2hzI''(0,1;2)], \tag{I19}$$

$$\sigma_1 = -\frac{a}{4(1-\nu)}[S_x\cos\phi+S_y\sin\phi][3I'(0,1;1)-z'I'(0,1;2)+(5-8\nu^2)I''(0,1;1)$$
$$-[h(3+4\nu)+z(3-4\nu)]I''(0,1;2)+2hzI''(0,1;3)], \tag{I20}$$

$$\sigma_{zz} = \frac{a}{4(1-\nu)}[S_x\cos\phi+S_y\sin\phi][(1-2\nu)I'(0,1;1)-z'I'(0,1;2)-(1-2\nu)I''(0,1;1)$$
$$+[h-z(3-4\nu)]I''(0,1;2)+2hzI''(0,1;3)], \tag{I21}$$

$$\sigma_2 = -\frac{a}{8(1-\nu)}[S\,e^{i\phi}\{(7-8\nu)I'(0,1;1)+(9-16\nu+8\nu^2)I''(0,1;1)$$
$$-z'I'(0,1;2)-(3-4\nu)z''I''(0,1;2)+2hzI''(0,1;3)\}$$
$$+\bar{S}e^{i3\phi}\{-H'(0,1;1)+(1-8\nu+8\nu^2)H''(0,1;1)-z'H'(0,1;2)$$
$$-z''(3-4\nu)H''(0,1;2)+2hzH''(0,1;3)\}], \tag{I22}$$

$$\tau_z = -\frac{a}{8(1-\nu)}[S\{4(1-\nu)I'(0,0;1)+4(1-\nu)I''(0,0;1)-z'I'(0,0;2)$$
$$-[h+z(3-4\nu)]I''(0,0;2)+2hzI''(0,0;3)\}$$
$$-\bar{S}e^{i2\phi}\{z'G'(0,0;2)+[h+z(3-4\nu)]G''(0,0;2)-2hzG''(0,0;3)\}]. \tag{I23}$$

Ring shear loading in the ρ and ϕ directions

$$u^c = \frac{a}{8\mu(1-\nu)}S_\rho e^{i\phi}[(3-4\nu)I'(1,1;0)-z'I'(1,1;1)+(5-12\nu+8\nu^2)I''(1,1;0)$$
$$-z''(3-4\nu)I''(1,1;1)+2hzI''(1,1;2)]+\frac{ia}{2\mu}S_\phi e^{i\phi}[I'(1,1;0)+I''(1,1;0)], \tag{I24}$$

$$w = -\frac{a}{8\mu(1-\nu)}S_\rho[z'I'(1,0;1)+4(1-\nu)(1-2\nu)I''(1,0;0)$$
$$+(3-4\nu)(z-h)I''(1,0;1)-2hzI''(1,0;2)], \tag{I25}$$

$$\sigma_1 = \frac{a}{4(1-\nu)} S_\rho [3I'(1,0;1) - z'I'(1,0;2) + (5-8\nu^2)I''(1,0;1) - [h(3+4\nu)+z(3-4\nu)]I''(1,0;2) + 2hzI''(1,0;3)]. \tag{I26}$$

$$\sigma_{zz} = -\frac{a}{4(1-\nu)} S_\rho [(1-2\nu)I'(1,0;1) - z'I'(1,0;2) - (1-2\nu)I''(1,0;1) + [h-z(3-4\nu)]I''(1,0;2) + 2hzI''(1,0;3)], \tag{I27}$$

$$\sigma_2 = \frac{a}{4(1-\nu)} S_\rho e^{i2\phi}[(3-4\nu)G'(1,0;1) - z'G'(1,0;2) + (5-12\nu+8\nu^2)G''(1,0;1) - (3-4\nu)z''G''(1,0;2) + 2hzG''(1,0;3)] + iaS_\varphi e^{i2\phi}[G'(1,0;1) + G''(1,0;1)], \tag{I28}$$

$$\tau_z = -\frac{a}{4(1-\nu)} S_\rho e^{i\phi}[2(1-\nu)I'(1,1;1) - z'I'(1,1;2) + 2(1-\nu)I''(1,1;1) - [h+z(3-4\nu)]I''(1,1;2) + 2hzI''(1,1;3)] - \frac{ia}{2} S_\varphi e^{i\varphi}[I'(1,1;1) + I''(1,1;1)]. \tag{I29}$$

A check on the analysis used to derive the above results was performed. This consisted of taking the Mindlin (1936) solution and integrating it around the circumference of a circle at a distance of h below the surface of an isotropic half space. The derivation was non-trivial and rather lengthy, however it provided results consistent with the above. This provides a check on the point force potentials in eqns (93) and (94), the elastic fields in eqns (97)–(102), (104)–(109), (111)–(116) and the isotropic limits of the sums (H1)–(H10) and the double sums in eqns (I1)–(I10).